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Thermal Response of a Thick Circular Plate With Internal Heat Sources in Time Fractional Frame

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Abstract

Present paper investigated the thermoelastic response of an axisymmetric two-dimensional time fractional thermoelastic problem. The order of the problem is $0 < \alpha \leq 2$ which occupy the space $D=\{(x,y,z)\in R^3:\ 0\leq r\leq b,\ -h\leq z\leq h\}$. Further, convection type boundaries with heating $Q_1\delta(t)\delta(r-r_0)$ and $Q_2\delta(t)\delta(r-r_0)$ are applied on the both surfaces respectively, whereas plate is subjected to the action of internal heat is the linear function of temperature. Next, integral transformation techniques are used to calculate temperature, displacement and thermal stresses. The graphical method is used to analyze the properties of Aluminum.

Keywords: Temperature distribution, Thermal stresses, Hankel transform, Caputo fractional derivative, circular plate.

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Introduction

Recently, growth of fractional order differential equations and integrals considerably reflected in formulation and description of many physical problems which are more beneficial than classical thermoelastic perspective. As it is known that for different physical situations microscopic level is quite essential but this ignored during processing by the classical Fourier law. The non-classical theory applied into the construction of equation for heat conduction and further, it requires thermoelastic model to improve the suitability of equation. The article investigated various materials and systems under different circumstances. The study successfully proposed uncoupled quasi-static theory of thermoelasticity with time-fraction derivative also it interpolates classical theory and thermoelasticity without energy dissipation as introduced by Green and Naghdi.1-D and 2-D equations were examined using Caputo derivative for fractional heat conduction [1]. Povstenko [2] investigated theory of thermal stresses with the Caputo time-fractional derivative of order α and he further, evaluated and examined the thermal stresses in an infinite body with a circular cylindrical hole by applying Laplace and Weber integral transforms.

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Caputo and Mainardi [3, 4]; Caputo [5], investigated and analyzed the relationship between the fractional derivative and theory of linear visco elasticity further they shown that the fractional model shows good agreement with experimental result. Povstenko [6] used integral transform technique to determine temperature distribution and thermal stresses in framework of uncoupled quasi-static uncoupled theory for an infinite medium with a spherical cavity. Mondal [7] used a vector-matrix differential equation in the Laplace transform domain and Fourier series expansion technique to develop a new theory of two-temperature generalized thermoelasticity in the context of a fractional order. Lata [8] found analytical solutions for displacement components, stresses, conductive temperature, temperature change and cubic dilatation for a homogeneous isotropic thick circular plate in the light of two temperature thermoelasticity theory in frequency domain. Povstenko [9] used integral transform technique to discuss effect of Robin boundaries by considering mathematical symmetric heat conduction equation of time-fractional order. Youssef [10] proved uniqueness theorem for a newly constructed fractional order thermoelasticity theory. Ezzat and Bary[11] developed mathematical modelling for perfect conducting materials within the context of fractional magneto-thermoelastic theory.

Kumar and Khobragade [12, 13,14, 15] studied and constructed mathematical modeling for various solid bodies due to the action of partially distributed heat supply and determined the thermal behaviour within the context of fractional order theory of thermoelasticity. Povstenko and Kyrylych[16] solved the fractional heat conduction equation for an infinite solid by using Laplace, Hankel and Fourier integral transforms with a penny-shaped crack in the case of axial symmetry under the prescribed heat flux loading at its surfaces. Roychoudhuri and Dutta [17] analyzed the thermoelastic interaction problem for an infinite solid with periodically distributed heat sources. Shaw and Mukhopadhyay, determined a thermoelastic problem of a functionally graded micro elongated medium with a periodically varying heat source[18].Very Recently, Lamba and Kamdi, discussed thermal behaviour of an axisymmetric problem of two-dimensional finite hollow cylinder with the fractional order derivative in which physical convection boundary conditions are assumed on the curved surface of cylinder. Also applies the integral transform method to analyses the temperature, thermal stresses and displacement [19].

In this present paper we extended the work done by Lamba et al. [19] to study thermoelastic response of an axisymmetric two-dimensional time fractional thermoelastic problem of a thick circular plate with the fractional order derivative of order $0 < \alpha \leq 2$ and boundary conditions were applied on upper and lower surface respectively. Here thick plate is subject to internal heat source, as heat is the linear function of the temperature. Further for numerical calculations Aluminum metal plate is considered and all the obtained results are depicted graphically employing simulating methods.

Mathematical Construction of The Problem

Let thickness of the thick circular plate is $2h$ with radius $r=b$, occupy the space $D= \{(x,y,z)\in R^3 : 0 \leq (x^2 +y^2)^{1/2} \leq b,\ -h \leq z \leq h\}$. Assume, circular plate is subject to internal heat source, heat is the function of temperature. Further, convection type boundaries with heating $Q_1\delta(t)\delta(r-r_0)$ and $Q_2\delta(t)\delta(r-r_0)$ are applied on the upper surface and lower surface resp., here δ represent the Dirac delta function. The material of the plate is assumed homogeneous and isotropic, properties of material remain uniform. To construct the above problem mathematically for nonlocal time fractional derivative of order α, we follow the fractional derivative of Caputo type as given by [20]

$$\frac{d^\alpha T(t)}{dt^\alpha}=\begin{cases}\dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^nT(\tau)}{d\tau^n}d\tau, \\ n-1<\alpha<n, \\ \dfrac{d^nT(\tau)}{dt^n}, \\ \alpha=n \end{cases} \quad (1)$$

with the Laplace transform method following as

$$L\left\{\frac{d^\alpha T(t)}{dt^\alpha}\right\}=s^\alpha L\{\overline{T}(s)\} -\sum_{k=0}^{n-1}T^{(k)}(0^+)s^{\alpha-1-k},\quad n-1<\alpha<n. \quad (2)$$

where; ‘s’ is transform parameter

Temperature distribution Function

By following [19], the governing transient heat conduction equation in transient form in context of Caputo type time fractional order parameter for a thick circular plate subjected to internal heat generation is given as follows

$$\kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta(r,z,t)}{\partial r}\right)+\frac{\partial^2\theta(r,z,t)}{\partial z^2}\right]+\Theta(r,z,t,\theta)=\frac{\partial^\alpha\theta(r,z,t)}{\partial t^\alpha} \quad (3)$$

Where $\theta(r,z,t)$ denotes temperature distribution function for the plate, $(\partial^\alpha\theta/\partial t^\alpha)$ represents the time fractional derivative of Caputo type with respect to time t, also internal heat source function is denoted here by $\Theta(r,z,t,\theta)$ and $\kappa=\lambda/\rho C$,

Where, λ is thermal conductivity of material,

ρ is density of material

C calorific capacity of the material.

By following [19], for sake of convenience we assume $\Theta(r,z,t,\theta)$ as the superimposition of the following simpler function given as

$$\Theta(r,z,t,\theta)=\Phi(r,z,t)+\psi(t)\theta(r,z,t) \quad (4)$$

and

$$T(r,z,t)=\theta(r,z,t)\exp\left[-\int_0^t\psi(\zeta)d\zeta\right] \quad (5)$$

$$\chi(r,z,t)=\Phi(r,z,t)\exp\left[-\int_0^t\psi(\zeta)d\zeta\right] \quad (6)$$

Next, for the purpose of simplicity $\chi(r,z,t)$ is taken as

$$\chi(r,z,t)=\frac{\delta(r-r_0)\delta(z-z_0)}{2\pi r_0}\delta(t), \quad 0\le r_0\le b,\ -h\le z_0\le h \quad (7)$$

On using (4) to (7) in the equation of heat conduction (3), we get

$$\kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(r,z,t)}{\partial r}\right)+\frac{\partial^2 T(r,z,t)}{\partial z^2}\right]+\chi(r,z,t)=\frac{\partial^\alpha T(r,z,t)}{\partial t^\alpha} \quad (8)$$

Here κ denotes the material's thermal diffusivity for thick circular plate.

Boundary Conditions

Following [21], the corresponding initial and boundary conditions for the assumed thermoelastic problem of thick circular plate are given as

$$T(r,z,t)=Q_0\,\delta(r-r_0)\,\delta(z-z_0)$$

at $t=0$, $0<\alpha\le 2$

for all $0\le r\le b,\ -h\le z\le h$ (9)

$$\frac{\partial T(r,z,t)}{\partial t}=0$$

at $t=0$, $1<\alpha\le 2$,

for all, $0\le r\le b,\ -h\le z\le h$ (10)

$$T(r,z,t)=0$$

at r = b

for all $-h\le z\le h,\ t>0$ (11)

$$\left[T(r,z,t)+k_1\frac{\partial T(r,z,t)}{\partial z}\right]_{z=h}=Q_1\,\delta(t)\,\delta(r-r_0)$$

for all $0\le r\le b$, $t>0$ (12)

$$\left[T(r,z,t)-k_2\frac{\partial T(r,z,t)}{\partial z}\right]_{z=-h}=Q_2\,\delta(t)\,\delta(r-r_0)$$

for all

$0\le r\le b$, $t>0$ (13)

Here, $\delta(r-r_0)$ and $\delta(z-z_0)$ are the Dirac Delta function having $0\le r_0\le b$, $-h\le z_0\le h$ respectively; $Q_1\delta(t)\delta(r-r_0)$ and $Q_2\delta(t)\,\delta(r-r_0)$ is the additional sectional heat applied on both sides of the plate at $z=h,-h$; also k_1 and k_2 represents the thermal conductivity coefficients. The equations (8) to (13) represent the mathematical construction of the problem with convective heat exchange boundary conditions under contemplation.

MATHEMATICAL MODELING

Heat conduction Analysis (Transient)

by Appling Hankel transform and its inverse of 'n' order over the variable 'r' to the equation (8) shows temperature distribution function under the boundary condition (11) obtained equation (14).

$$T^*(\beta_n, z, t) = \int_0^b T(r, z, t)\, r\, k_0(\beta_n, r)\, dr,$$

$$T(r, z, t) = \sum_{n=1}^{\infty} T^*(\beta_n, z, t)\, k_0(\beta_n, r) \qquad (14)$$

Where,

$k_0(\beta_n, r) = \frac{\sqrt{2}}{b}\left(\frac{J_0(\beta_n r)}{J_1(\beta_n b)}\right)$ and the eigen values $\beta_1, \beta_2, \beta_3 ....$ are the roots of characteristic equation $J_0(\beta b) = 0$, and J_n is the Bessel function of the first kind of order *n*.

Where $T^*(\beta_n, z, t)$ denotes the Hankel transform of $T(r, z, t)$ in the transformed domain.

Secondly, define the finite domain Marchi-Fasulo integral transform of $T(z)$ in the range $-h \le z \le h$ as in [22] that corresponds to the boundary conditions of type (12)-(13) as

$$\overline{T}(m, t) = \int_{-h}^{h} T(z, t) P_m(z)\, dz,$$

$$T(z, t) = \sum_{m=1}^{\infty} \frac{\overline{T}(m, t)}{\lambda_m} P_m(z) \qquad (15)$$

Where, $\overline{T}(m, t)$ denotes the Marchi-Fasulo integral transform of $T(z, t)$ in the transformed field, and the core is given by the orthogonal functions in the interval $-h \le z \le h$ as

$$P_m(z) = Q_m \cos(a_m z) - W_m \sin(a_m z),$$

where

$$Q_m = a_m(k_1 + k_2)\cos(a_m h),$$

$$W_m = 2\cos(a_m h) + (k_2 - k_1) a_m \sin(a_m h),$$

$$\lambda_m = \int_{-h}^{h} P_m^2(z)\, dz = h[Q_m^2 + W_m^2] + \frac{\sin(2a_m h)}{2a_m}[Q_m^2 - W_m^2]$$

The roots of characteristic equation are the eigen values a_m.

$$[k_1 a \cos(ah) + \sin(ah)] \times [\cos(ah) + k_2 a \sin(ah)] = [k_2 a \cos(ah) - \sin(ah)] \times [\cos(ah) - k_1 a \sin(ah)]$$

Applying the above defined transformation rules ((14) and (15)) to equation (8) under the condition (11)-(13), the following reduction is made

$$\frac{d^\alpha \overline{T}^*}{dt^\alpha} + \kappa\left(\alpha_m^2 + \beta_n^2\right)\overline{T}^* = H(\alpha_m, \beta_n) \qquad (16)$$

with transformed initial conditions as

$$\overline{T}^*(\alpha_m, \beta_n, t) = Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0)$$

at $t = 0$, $0 < \alpha \le 2$ (17)

$$\frac{\partial \overline{T}^*(\alpha_m, \beta_n, t)}{\partial t} = 0$$

at $t = 0$, $1 < \alpha \le 2$ (18)

where

$$H(\alpha_m, \beta_n) = \left\{ \begin{array}{l} \frac{Q_1 P_m(h) \kappa r_0}{k_1} - \\ \frac{Q_2 P_m(-h) \kappa r_0}{k_2} + \\ \frac{P_m(z_0)}{2\pi} \end{array} \right\} k_0(\beta_n, r_0)$$

Now, by applying Laplace transform w.r.t. 't' and their corresponding inverse to the equation (16) by using corresponding initial conditions defined in (17) and (18) one obtains

$$\overline{T}^*(\alpha_m, \beta_n, t) = \left\{ \begin{array}{l} E_\alpha(-\kappa\left(\alpha_m^2 + \beta_n^2\right)t^\alpha) \times \\ Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{array} \right. + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2 + \beta_n^2\right)t^\alpha)\, H(\alpha_m, \beta_n) \Big\} \qquad (19)$$

Next, by taking inverse of Marchi-Fasulo and Hankel integral transform to the equation (19), one obtains the temperature solution as

$$T(r, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ \begin{array}{l} E_\alpha(-\kappa\left(\alpha_m^2 + \beta_n^2\right)t^\alpha) \\ \times Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \\ + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2 + \beta_n^2\right)t^\alpha) \times \\ H(\alpha_m, \beta_n) \end{array} \right\} \times \frac{P_m(z)}{\lambda_m} k_0(\beta_n, r) \qquad (20)$$

Further, taking into account of the first equation (4), the final temperature distribution function is shown by

$$\theta(r,z,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\begin{cases}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\\ Q_0\, r_0\, k_0(\beta_n,r_0)\,P_m(z_0)\end{cases} \\ +t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\Big\} \\ \frac{P_m(z)}{\lambda_m}k_0(\beta_n,r)\exp\left[\int_0^t\psi(\zeta)d\zeta\right] \quad (21)$$

Hence, the equation (21) shows, the temperature distribution at each instant of time with fractional order derivate of order α and over the thick circular plate of fixed height when there are convection type boundaries on both sides

Displacements and thermal stresses Formulation

The expression of the Navier's equations in the absence of body forces for the axisymmetric problem of two-dimensional finite circular plate can be expressed [23] as

$$\nabla^2 u_r-\frac{u_r}{r}+\frac{1}{1-2\upsilon}\frac{\partial e}{\partial r}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial r}=0 \quad (22)$$

$$\nabla^2 u_z-\frac{1}{1-2\upsilon}\frac{\partial e}{\partial z}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial z}=0 \quad (23)$$

In equation (22) and (23), u_z and u_r represent the displacement components along the axial and radial direction respectively. The Dilatation *e* is expressed as

$$e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z} \quad (24)$$

In cylindrical coordinate the radial and axial displacement function are expressed in terms of thermoelastic Goodier's displacement potential $\phi(r,z,t)$ and Michell's function M as [23]

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial^2 M}{\partial r\partial z} \quad (25)$$

$$u_z=\frac{\partial\phi}{\partial z}+2(1-\upsilon)\nabla^2 M-\frac{\partial^2 M}{\partial^2 z} \quad (26)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\theta \quad (27)$$

Here thermoelastic Goodier's potential function has to satisfy the above equation.

$$\nabla^2(\nabla^2 M)=0 \quad (28)$$

Equation (28) should satisfies by Michell's function M.

where $\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}$

The stress function $\sigma_{rr},\sigma_{\theta\theta},\sigma_{zz},\sigma_{rz}$ are expressed by using the potential function ϕ & Michell's function *M* as

$$\sigma_{rr}=2G\left\{\begin{array}{l}\left(\frac{\partial^2\phi}{\partial r^2}-\nabla^2\phi\right)+\\ \frac{\partial}{\partial z}\left(\upsilon\nabla^2 M-\frac{\partial^2 M}{\partial r^2}\right)\end{array}\right\} \quad (29)$$

$$\sigma_{\theta\theta}=2G\left\{\begin{array}{l}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}-\nabla^2\phi\right)+\\ \frac{\partial}{\partial z}\left(\upsilon\nabla^2 M-\frac{1}{r}\frac{\partial M}{\partial r}\right)\end{array}\right\} \quad (30)$$

$$\sigma_{zz}=2G\left\{\begin{array}{l}\left(\frac{\partial^2\phi}{\partial z^2}-\nabla^2\phi\right)+\\ \frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}\right)\end{array}\right\} \quad (31)$$

$$\sigma_{rz}=2G\left\{\begin{array}{l}\frac{\partial^2\phi}{\partial r\partial z}+\\ \frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}\right)\end{array}\right\} \quad (32)$$

Where G is shear modulus and υ is Poisson ratio. For the traction free surface at inner and outer radii the expression for σ_{rr} is

$$\sigma_{rr}\big|_{r=b}=\sigma_{rz}\big|_{r=b}=0 \quad (33)$$

The Solution of Displacements and thermal stresses

Equation (34) used to determine Goodier's thermoelastic displacement potential function ϕ. The temperature distribution function given in equation (21) to the equation (27) was used to generate equation (34)

$$\phi(r,z,t) = -\left(\frac{1+\upsilon}{1-\upsilon}\right) a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\left(\alpha_m^2+\beta_n^2\right)} \begin{Bmatrix} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha) \\ \times Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{Bmatrix} + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\, H(\alpha_m,\beta_n)\Big\} \frac{P_m(z)}{\lambda_m} k_0(\beta_n, r)\exp\left[\int_0^t \psi(\zeta)d\zeta\right] \quad (34)$$

Likewise, assumed, solution of Michell's function M for equation (28) ;

$$M(r,z,t) = -\left(\frac{1+\upsilon}{1-\upsilon}\right) a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\left(\alpha_m^2+\beta_n^2\right)} \begin{Bmatrix} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha) \\ \times Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{Bmatrix} + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\, H(\alpha_m,\beta_n)\Big\}\times k_0(\beta_n, r)\exp\left[\int_0^t \psi(\zeta)d\zeta\right]\times [\cosh(\beta_n z) + z\sinh(\beta_n z)] \quad (35)$$

Next, to find the displacement components expression, by using the ϕ and M obtained from equations (34), (35) and it is inserted to equations (25) and (26) obtains equation (37)

$$u_r = -\left(\frac{1+\upsilon}{1-\upsilon}\right) a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\left(\alpha_m^2+\beta_n^2\right)} \begin{Bmatrix} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{Bmatrix} + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\, H(\alpha_m,\beta_n)\Big\}\times \left(\frac{\sqrt{2}\,\beta_n\, J_1(\beta_n r)}{b\, J_1(\beta_n b)}\right)\exp\left[\int_0^t \psi(\zeta)d\zeta\right] \times \{-Q_m\cos(a_m z) + W_m \sin(a_m z) + z\,\beta_n\cos(\beta_n z) + \sin(\beta_n z) + \beta_n \sinh(\beta_n z)\} \quad (36)$$

$$u_z = -\left(\frac{1+\upsilon}{1-\upsilon}\right) a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\left(\alpha_m^2+\beta_n^2\right)} \begin{Bmatrix} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{Bmatrix} + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\, H(\alpha_m,\beta_n)\Big\}\times k_0(\beta_n, r)\exp\left[\int_0^t \psi(\zeta)d\zeta\right] \times \{a_m[Q_m \sin(a_m z) + W_m\cos(a_m z)] + \beta_n[-2+\beta_n+4\upsilon]\cosh(\beta_n z) + z\,\beta_n^2 \sinh(\beta_n z)\} \quad (37)$$

Thus, operating the two-displacement component, the dilation is recognized as

$$e = -\left(\frac{1+\upsilon}{1-\upsilon}\right) a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\left(\alpha_m^2+\beta_n^2\right)} \begin{Bmatrix} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{Bmatrix} + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\, H(\alpha_m,\beta_n)\Big\}\times k_0(\beta_n, r)\exp\left[\int_0^t \psi(\zeta)d\zeta\right] \times \{(\beta_n^2 + a_m^2)[Q_m\cos(a_m z) - W_m \sin(a_m z)] + 2(-1+2\upsilon)\beta_n^2\sinh(\beta_n z)\} \quad (38)$$

Next, the components of stresses are obtained by substituting the values of ϕ from equation (34) and M from equation (35) in equation (29)-(32), one obtains

$$\sigma_{rr} = 2G\left(\frac{1+\upsilon}{1-\upsilon}\right) a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{\left(\alpha_m^2+\beta_n^2\right)} \begin{Bmatrix} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n, r_0)\, P_m(z_0) \end{Bmatrix} + t^{\alpha-1} E_\alpha(-\kappa\left(\alpha_m^2+\beta_n^2\right)t^\alpha)\, H(\alpha_m,\beta_n)\Big\}\times \frac{1}{\sqrt{2}\, b\, J_1(b\beta_n)}\exp\left[\int_0^t \psi(\zeta)d\zeta\right] \times\{-\beta_n^2\, J_2(r\beta_n)(-Q_m\cos(a_m z) + W_m\sin(a_m z) + z\beta_n\cosh(\beta_n z) + (1+\beta_n)\sinh(\beta_n z)$$

$$+J_0(r\beta_n)[z\beta_n^3\cosh(\beta_n z)+ (2a_m^2+\beta_n^2)(Q_m\cos(a_m z)- W_m\sin(a_m z)) + \beta_n^2(1+\beta_n+4\upsilon)\sinh(\beta_n z)]\} \quad (39)$$

$$\sigma_{\theta\theta}=2G\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{Bmatrix}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times\\ Q_0\, r_0\, k_0(\beta_n,r_0)\,P_m(z_0)\end{Bmatrix}$$

$$+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}\exp\left[\int_0^t\psi(\zeta)d\zeta\right]$$

$$\times\{[\beta_n^2 J_2(r\beta_n)[-Q_m\cos(a_m z)+ W_m\sin(a_m z)+z\beta_n\cosh(\beta_n z)+ (1+\beta_n)\sinh(\beta_n z)]$$

$$+J_0(\beta_n r)[z\beta_n^3\cosh(\beta_n z)+ (2a_m^2+\beta_n^2)[Q_m\cos(a_m z)- W_m\sin(a_m z)] \quad (40)$$

$$\sigma_{zz}=2G\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{Bmatrix}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times\\ Q_0\, r_0\, k_0(\beta_n,r_0)\,P_m(z_0)\end{Bmatrix}$$

$$+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)H(\alpha_m,\beta_n)\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}\exp\left[\int_0^t\psi(\zeta)d\zeta\right]$$

$$\times\{[\beta_n^2 J_2(r\beta_n)[Q_m\cos(a_m z)- W_m\sin(a_m z)]+J_0(r\beta_n)\times [-2z\beta_n^3\cosh(\beta_n z)]$$

$$+(2a_m^2+\beta_n^2)[Q_m\cos(a_m z) -W_m\sin(a_m z)] -2\beta_n^2(5+\beta_n-4\upsilon)\sinh(\beta_n z)\} \quad (41)$$

and

$$\sigma_{rz}=2G\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{Bmatrix}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times\\ Q_0\, r_0\, k_0(\beta_n,r_0)\,P_m(z_0)\end{Bmatrix}$$

$$\left.\begin{matrix}+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times\\ H(\alpha_m,\beta_n)\end{matrix}\right\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}\exp\left[\int_0^t\psi(\zeta)d\zeta\right]$$

$$\times\{\sqrt{2}\,\beta_n\,J_1(r\beta_n)[a_m W_m\cos(a_m z) +a_m Q_m\sin(a_m z)+\beta_n(\beta_n+2\upsilon)\times \cosh(\beta_n z)+z\beta_n^2\sinh(\beta_n z)]\}, \quad (42)$$

SPECIAL CASE

On Setting

$$\psi(\zeta)=\delta(\zeta-\zeta_0),\quad 0<\zeta_0<t \quad (43)$$

$$\Rightarrow\ \int_0^t\psi(\zeta)\,d\zeta=1 \quad (44)$$

Putting the value of equation (44) into equations (21) and (39) to (42), equation (45) determines the temperature and stresses on the surfaces.

$$\theta(r,z,t)=e\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\begin{Bmatrix}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times\\ Q_0\, r_0\, k_0(\beta_n,r_0)\,P_m(z_0)\end{Bmatrix} +t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\}\times \frac{P_m(z)}{\lambda_m}k_0(\beta_n,r) \quad (45)$$

$$\sigma_{rr}=2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{Bmatrix}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times\\ Q_0\, r_0\, k_0(\beta_n,r_0)\,P_m(z_0)\end{Bmatrix}$$

$$+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}$$

$$\times\{-\beta_n^2 J_2(r\beta_n)(-Q_m\cos(a_m z)+ W_m\sin(a_m z)+z\beta_n\cosh(\beta_n z)+(1+\beta_n)\sinh(\beta_n z)$$

$$+J_0(r\beta_n)[z\beta_n^3\cosh(\beta_n z)+(2a_m^2+\beta_n^2)(Q_m\cos(a_m z)- W_m\sin(a_m z))+\beta_n^2(1+\beta_n+4\upsilon)\times \sinh(\beta_n z)]\}, \quad (46)$$

$$\sigma_{\theta\theta}=2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{cases}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n,r_0)\, P_m(z_0)\end{cases}$$

$$+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}$$

$$\times\{[\beta_n^2 J_2(r\beta_n)[-Q_m\cos(a_m z)+ W_m\sin(a_m z)+z\beta_n\cosh(\beta_n z)+(1+\beta_n)\sinh(\beta_n z)]$$

$$+J_0(\beta_n r)[z\beta_n^3\cosh(\beta_n z)+(2a_m^2+\beta_n^2)[Q_m\cos(a_m z)- W_m\sin(a_m z)]+\beta_n^2(1+\beta_n+4\upsilon)\times \sinh(\beta_n z)]\}, \quad (47)$$

$$\sigma_{zz}=2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{cases}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n,r_0)\, P_m(z_0)\end{cases}$$

$$+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}$$

$$\times\{[\beta_n^2 J_2(r\beta_n)[Q_m\cos(a_m z)-W_m\sin(a_m z)] +J_0(r\beta_n)[-2z\beta_n^3\cosh(\beta_n z)]$$

$$+(2a_m^2+\beta_n^2)[Q_m\cos(a_m z)-W_m\sin(a_m z)] -2\beta_n^2(5+\beta_n-4\upsilon)\sinh(\beta_n z)\} \quad (48)$$

and

$$\sigma_{rz}=2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)a_t \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(\alpha_m^2+\beta_n^2)}\begin{cases}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\times \\ Q_0\, r_0\, k_0(\beta_n,r_0)\, P_m(z_0)\end{cases}$$

$$+t^{\alpha-1}E_\alpha(-\kappa(\alpha_m^2+\beta_n^2)t^\alpha)\,H(\alpha_m,\beta_n)\}\times \frac{1}{\sqrt{2}\,b\,J_1(b\beta_n)}$$

$$\times\{\sqrt{2}\,\beta_n J_1(r\beta_n)[a_m W_m\cos(a_m z)+ a_m Q_m\sin(a_m z)+\beta_n(\beta_n+2\upsilon)\times \cosh(\beta_n z), \quad (49)$$

NUMERICAL CALCULATIONS

For the numerical computations, one can assume material properties of Aluminum metal for a thick circular plate with non-dimensional variables as shown below:

$$\bar{\theta}=\frac{\theta}{\theta_R},\ \eta=\frac{r}{a},\quad \xi=\frac{z}{a},\ \ \tau=\frac{\kappa t}{a^2},$$

$$\bar{h}=\frac{h}{a},\ (u_r,u_z)=\frac{(\bar{u}_r,\bar{u}_z)}{a\alpha_t\theta_R}$$

$$(\bar{\sigma}_{rr},\bar{\sigma}_{\theta\theta},\bar{\sigma}_{zz},\bar{\sigma}_{rz})=\frac{(\sigma_{rr},\sigma_{\theta\theta},\sigma_{zz},\sigma_{rz})}{E\alpha_t\theta_R}$$

Modulus of Elasticity: E (dynes/cm^2) = 6.9×10^{11}, Shear modulus: G (dynes/cm^2) = 2.7×10^{11}, Poisson ratio: $\upsilon = 0.281$, Coefficient of thermal expansion: α_t(cm/cm-^{0}C) = 25.5×10^{-6},

Thermal diffusivity, κ (cm^2/ s^α) = 0.86,
Thermal conductivity, λ (cal-cm/^{0}C/sec/ cm^2) = 0.48,
Outer radius, b (cm) = 3,
Thickness, h (cm) = 1, r_0 (cm) $=1.5$, z_0 (cm) $=0.5$

GRAPHICAL ANALYSIS

$\alpha=0.5$, $\alpha=1$, $\alpha=1.5$, $\alpha=2$ are the parameters for temperature and thermal stresses for fractional order. MATHEMATICA software used to obtain the various stresses acted on the circular plate due to the heat. Figure 1 to figure 5 shows the graphical representation of the heat applied on the both sides of the circular plated obtained from equation (45) to equation (49). For considerable mathematical simplicities in the foregoing analysis we set the coefficients, $k_1 = 0.86$ and $k_2 = 1$.

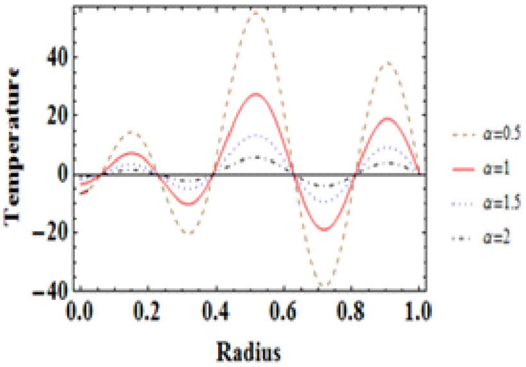

Figure 1: Temperature distribution along the radial direction

Figure 1, demonstrated the temperature distribution along the radius of the plate. The figure 1 shows the optimum temperature at fraction order parameter of value α=1 against the other value of α=2, 1.5 and 0.5. This is due to the increase in thickness of the plate.

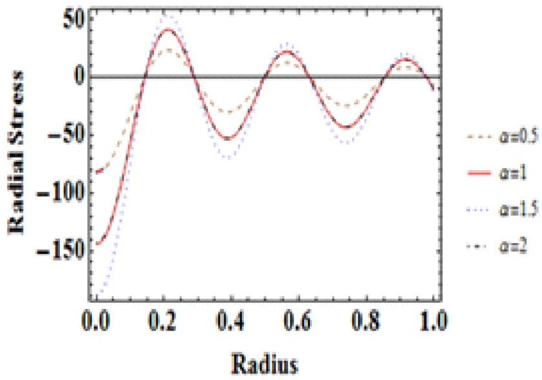

Figure 2: Radial stress distribution along the radial direction

Figure 2, displays optimal radial stresses at α = 1 and found to be more at inner radius and lower at outer radial edge and then terminated to zero at outer radial ends. This is because of traction free boundary conditions.

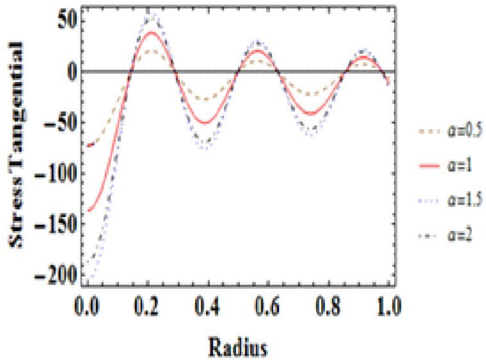

Figure 3: Tangential stress distribution along the radial direction

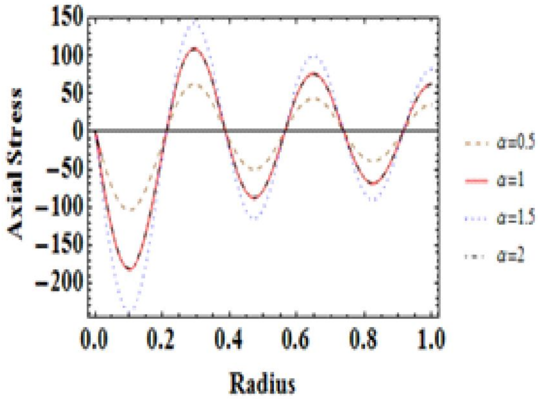

Figure 4: Axial stress distribution along the radial direction

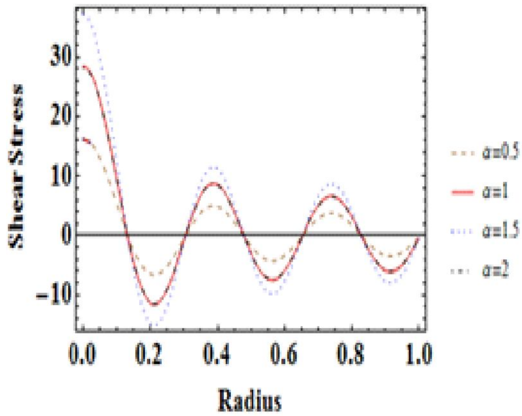

Figure 5: Shear stress distribution along the radial direction

Figure 3, 4 and 5, shows graphical representations of the distribution of tangential, axial and shear stresses along the radial stresses respectively for $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$, and $\alpha = 2$. Initially variation of stresses found increases in the tangential, zero in axial and decreases in shear case near to the inner region and then it becomes sinusoidal towards the outer radii. But variation of the speed of propagation of the thermal signals is found significantly different for the different values of the fractional-order parameter α for all the tangential, axial and shear cases. Further, prescribed mathematical traction free boundary condition satisfies at the outer radial end.

Conclusion

The mathematical model was developed to obtain the temperature distribution along the circular plate using the finite Hankel transform, finite Marchi-Fasulo and Laplace transform methods. Also, convection type boundaries with heating $Q_1\delta(t)\delta(r-r_0)$ and $Q_2\delta(t)\delta(r-r_0)$ on both surfaces.

Proposed model satisfying the required conditions to solve the thermoelastic problems of any kind by fractional order derivatives. the model also being useful for to examine the parameters like thermal conductivity, thermal stresses and the temperature distribution along the any circular surfaces. This article proposes new methodology for designing the mathematical model for analysis of various kind of surfaces for engineering applications.

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