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On Certain Triple Integral Relations Involving Elementary Functions

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Abstract

The aim of this paper is to establish three triple integral relations involving elementary functions. A number of triple integrals can be deduced by proper specialization of the unknown functions f and g occurring in these relations. For the sake of illustration, one of our integral relations is applied to evaluate a general triple integral involving Asgar, Gautam and Goyal multivariable A- function.

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Introduction

Many authors have worked on the problem of obtaining integral relations involving higher classes of special functions of one and more variables.[4,6] In this paper we derive three new integral relations associated with some elementary functions and illustrate how they can be applied to derive triple integrals which may be of interest.

Integral Relations

$$\int_0^\infty \int_0^\infty \int_0^\infty (x^2+y^2)^{-1/2} \, exp\, [\, i\, (x^2+y^2+z^2) \left(\frac{x^2-y^2}{x^2+y^2}\right)].$$

$$\cos\, [2n\, tan^{-1}\, (y/x)]\, f\, (x^2+y^2+z^2)\, .\, g\, \{\, tan^{-1} \left(\frac{x^2+y^2}{2}\right)\}\, dx\, dy\, dz$$

$$= \frac{\pi i n}{2} \int_0^\infty \int_0^\infty J_n(u^2+v^2) f\, (u^2+v^2)\, g\, \{tan^{-1} \left(\frac{v}{u}\right)\text{dudv} \qquad (2.1)$$

where n is any integer, positive or negative and the functions f and g are so constrained that the various integrals involved in (2.1) exist.

$$\int_0^\infty \int_0^\infty \int_0^\infty (xy)^{v+1/2} (x^2+y^2)^{-(v+1)}\, exp\, [\, i\, (x^2+y^2+z^2) \left(\frac{x^2-y^2}{x^2+y^2}\right) \cos\varphi].$$

$$J_{v-1/2}[\frac{2xy}{x^2+y^2}(x^2+y^2+z^2)\sin\varphi] C_n^v \left(\frac{x^2-y^2}{x^2+y^2}\right) f\, (x^2+y^2+z^2) g[\, tan^{-1} \left(\frac{x^2+y^2}{2}\right)]\text{dx dy dz}$$

$$= 2^{-(v+1)} \sqrt{\pi i n}\, (\sin\varphi)^{v-1/2}\, C_n^v\, (\cos\varphi) \int_0^\infty \int_0^\infty (u^2+v^2)^{-1/2}\, J_{v+n}\, (u^2+v^2).$$

$$f\, (u^2+v^2)\, g\, [tan^{-1} \left(\frac{v}{u}\right)]\text{dudv} \qquad (2.2)$$

provided that Re (v) >-1/2, n = 0,1, 2, . . . and f and g are so constrained that the various integrals involved in (2.2) exist.

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$$\int_0^\infty \int_0^\infty \int_0^\infty (xy)^{2v} (x^2+y^2)^{-(2v+1/2)}\, W^{-v}\, Z_v\, (W)\, C_n^v \left(\frac{x^2-y^2}{x^2+y^2}\right).$$

$$f\, (x^2+y^2+z^2) g\, [\, tan^{-1} \left(\frac{x^2+y^2}{2}\right)]\text{dx dy dz}$$

$$= \frac{2^{3v}\, \pi\, \Gamma(n+2v) Z_{v+n}\, (t)\, t^{-v}}{n!\, \Gamma(v)} \int_0^\infty \int_0^\infty (u^2+v^2)^{-v}\, J_{v+n}\, (u^2+v^2)\, .$$

$$f\, (u^2+v^2)\, g\, [tan^{-1} \left(\frac{v}{u}\right)]\text{dudv} \qquad (2.3)$$

valid under the same conditions as those stated for (2.2) above.

In (2.1), (2.2) and (2.3) $J_v(x)$ is the Bessel function of the first kind, $C_x^v\, (x)$ is the Gegenbauer polynomial and $Z_v(x)$ stands for any Bessel function of the first, second or third kind. Also

$$W = [(x^2+y^2+z^2)^2 + t^2 - 2t \left(\frac{x^2-y^2}{x^2+y^2}\right) [(x^2+y^2+z^2)]^{-1/2} \qquad (2.4)$$

Proof of (2.1): We have from[7]

$$J_n(x) = \frac{1}{\pi i^n}\int_0^{\pi} exp\ (iz\cos\theta)\cos n\theta\ \ d\theta \qquad (2.5)$$

In order to derive the integral relations (2.1), we replace x by $r^2\theta$ by 2θ in (2.5), multiply both sides by rf $(r^2)\ g(\emptyset)dr\ d\emptyset$ and then integrate the resulting equation with respect to r and $\emptyset$ over the intervals $(0,\infty)$ and $(o,\pi/2)$, respectively. we thus get

$$\frac{\pi i^n}{2}\int_0^{\infty}\int_0^{\pi/2}\{J_n(r^2)\ rf(r^2)g(\emptyset)dr\}\ d\emptyset$$

$$= \int_0^{\infty}\int_0^{\pi/2}\int_0^{\pi/2}\exp(ir^2\cos 2\theta)\cos 2n\theta\ \frac{r\sin\emptyset}{r\sin\emptyset}\ rf\ r^2\ ()g(\emptyset)drd\emptyset\ d\theta \qquad (2.6)$$

If we make the substitution x = r sin $\emptyset$ $\cos\theta$, y = r sin $\emptyset$ $\cos\theta$ and z = r cos $\emptyset$ on right – hand side of (2.6) and set u = r cos $\emptyset$, v = r sin $\emptyset$ in left hand side, we are easily led to the integral relation (2.1).

To prove the integral relations (2.3) and (2.4), we start with the known integrals[3] and proceed on the lines similar to those mentioned in the proof of (2.1).

Useful Deduction

The function g appearing in our integral (2.1) (2.2) and (2.3) may be chosen appropriately to derive various triple integrals. For example, if in (2.1), we get

$$g(t) = \cos 2(\mu t)(\sin t)^{\vartheta} \qquad (3.1)$$

and simplify the right – hand side of the resulting equation by means of a known integral relation,[6] we arrive at the following result:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^2+y^2)^{(v-1)/2}(x^2+y^2+z^2)^{v/2}\quad \exp[\,i\,(x^2+y^2+z^2)$$

$$\left(\frac{x^2-y^2}{x^2+y^2}\right)]\cos\left[2n\left(\tan^{-1}\frac{y}{x}\right)\cos\left[2\mu\left\{tan^{-1}\left(\frac{x^2+y^2}{2}\right)\right\}\right] f(x^2+y^2+z^2)dx\,dy\,dz$$

$$= \frac{\sqrt{\pi}\, i^n\, \Gamma\left(\frac{1}{2}\pm\mu\right)\Gamma\left(\frac{1+v}{2}\right)\Gamma(1+v/2)}{8\,\Gamma\,(1+v/2\pm\mu)}\int_0^{\infty} f(t)J_n\,(t)dt \qquad (3.2)$$

Re (v) > 0, n is an integer, positive or negative and f is so chosen that the integrals on both both sides of (3.2) exist. Now in (3.2), we get

$$f\,(t) = t^{\lambda-1}\,H\,[z_1t^{\rho_1},\ldots,z_rt^{\rho_r}]$$

where

$$A\,[z_1,\ldots,z_r] = A^{m,n;\,m_1,n_1;\ldots;\,m_r,n_r}_{p,q;\,p_1,q_1;\ldots;\,p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\left|\begin{matrix}(a_j;A_j',\ldots,A_j^{(r)})_{1,p};\ (c_j',C_j'')_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ (b_j;B_j',\ldots,B_j^{(r)})_{1,q};\ (d_j';D_j'')_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

is a special case of the multivariable A- function due to Asgar, Gautam and Goyal.[1,2]

Evaluating the resulting integral with the help of a known integral,[5] we arrive at the following interesting triple integral which is believed to be new:

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} x^2+y^2)^{\lambda-(v+2)/2}(x^2+y^2)^{(v-1)/2}\quad \exp[\,i\,(x^2+y^2+z^2)\left(\frac{x^2-y^2}{x^2+y^2}\right)].$$

$$\cos\left[2n\left(\tan^{-1}\frac{y}{x}\right)\right]\cos\{2\mu(tan^{-1}\left(\frac{x^2+y^2}{2}\right)\}.$$

$$A\,[\,Z_1(x^2+y^2+z^2)^{\rho_1},\ldots,Z_r(x^2+y^2+z^2)^{\rho_r}\ dx\,dy\,dz$$

$$= \frac{2^{\lambda-4}\sqrt{\pi}\, i^n\, \Gamma\left(\frac{1}{2}\pm\mu\right)\Gamma\left(\frac{1+v}{2}\right)\Gamma(1+v/2)}{\Gamma\,(1+v/2\pm\mu)}\,.$$

$$A^{M,N;\,M_1,N_1;\ldots;\,M_r,N_r}_{P+2,Q;\,P_1,Q_1;\ldots;\,P_r,Q_r}\left[\begin{matrix} Z_1 2^{\rho_1} \\ Z_r 2^{\rho_r}\end{matrix}\left|\begin{matrix}\left(1\pm\frac{n}{2}-\frac{\lambda}{2};\frac{\rho_r}{2},\ldots,\frac{\rho_r}{2}\right)(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P};\ (c_j',\gamma_j^{(\prime)})_{1,P_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,P_r} \\ (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q};\ (d_j';\delta_j')_{1,Q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,Q_r}\end{matrix}\right.\right] \quad (3.3)$$

provided that Re $(v) > 0, \rho_i > 0\ (i = 1,\ldots,r)$, n is an integer, positive or negative,

$$Re\,(\lambda) + \sum_{i=1}^{r}[\rho_i \min_{1\le j\le m_i} Re\,(d_j^{(i)}/\delta_j^{(i)})] + n > 0,$$

$$Re\,(\lambda) + \sum_{i=1}^{r}[\rho_i \max_{1\le j\le n_i} Re\,(c_j^{(i)}/\gamma_j^{(i)})] - \frac{3}{2} < 0$$

$$\eta_i = -\sum_{j=1}^{P}\alpha_j^{(i)} - \sum_{j=1}^{Q}\beta_j^{(i)} + \sum_{j=1}^{N_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{P_i}\gamma_j^{(i)} + \sum_{j=1}^{M_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{Q_i}\delta_j^{(i)} > 0$$

and $|\arg(z_k)| < \frac{1}{2}\eta_i\,\pi, (i = 1,\ldots,r).$

The triple integral (3.3) is quite general in character due to general nature of the multivariable A- function involved therein. Thus, by appropriately reducing this multivariable A- function in terms of simpler special functions, one can easily obtain a considerably large number of triple integrals to mathematical analysis and applied mathematicians.

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