RESEARCH ARTICLE

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Study of Convective Heating in a Thermosensitive Functionally Graded Plate with Time-Fractional Heat Equation

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Abstract

The proposed study emphasis on to examine the temperature and stress distribution on functionally graded thermosensitive rectangular plate. Caputo derivative of order 0 to 2 is applied to analyses the rectangular plate when subjected to the convection heat.

Present article illustrated the methodology and the mathematical model opt for framing the expression to address the issues regarding the properties of ceramic based material affected by the temperature. The present article studied the Goodier's displacement function and Boussinesq Harmonic functions for developing the mathematical model for the rectangular plate and other engineering applications.

Keywords: Convective Heating, Functionally Graded Material, Rectangular Plate, Thermosensitive.

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Introduction

Various researchers in past worked on the Caputo derivatives in order to develop the new mathematical models for different geometries to analyses the effects of temperature on the functionally graded plate. By applying pertuburation method properties of the material was analysed at various temperature [1] . Popovych [2,3], Popovych and Makhorkin [4] and Kushnir and Popovych. [5] studied the thermal affect in various bodies due to thermosensitivity. Further the study extended to analyses temperature and stress effect on the ceramic material for on-dimensional geometry [6]. Povstenko [7] studied quasi-static approach under uncoupled thermoelasticity when the material subjected to the conductive heat [7]. Povstenko [8], Kushnir and Popovych [9] and Popovych et. al. [10] investigated stress functions by considering Caputo type heat equation of time fraction order by using Integral transformation for an infinite body. Povstenko [11] Applied integral transformation in order to examine the time fractional diffusion equation for infinite cylinder. Povstenko [12-13, 14] determined solution in cylindrical coordinates of a non-axisymmetric problem under time-fractional wave equation with a heat source in an infinite medium by using Hankel and Fourier transform. Kedar et al. [15, 16-17] determined thermal behavior in nonhomogeneous plate with material dependent properties and studied its deformation analysis.

The hollow cylinder with uniform temperature were analyzed using integral transform [18]. The study evident of new equation for displacement stress function. Kumar and Kumar [19] did thermoelastic analysis of a beam with diffusion by Eigen value approach. the Axisymmetric problem of 2-D was studied to obtain the temperature and stresses on

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cylinder [20] The integral transformation applied to investigate heat transfer behavior (time fractional) for a non-homogeneous thick hollow cylinder subjected to heat [21]. Lamba and Deshmukh [22] studied the problem with hygrothermal effect in a finite solid cylinder and successfully determined its thermal stress and temperature function. Kamdi and Kumar [23] discussed behaviour of stress and temperature variation in a fin in theory of fractional thermoelasticity. Thakre and Warbhe [24-25] analyzed deflection and temperature distribution effect on material (functionally graded) with heat source using fractional order theory.

The study focused on determination of distribution of temperature and stress in case of temperature dependent functionally graded fractional order thermosensitive rectangular plate by using modified integral transformation technique. For the purpose of numerical analysis a ceramic- based functionally graded material is considered and the results of temperature distribution and associated stresses are presented graphically for both homogeneous and nonhomogeneous case.

Caputo Fractional Derivative

The Caputo derivative for the function $G(t)$ is defined as;

$$\frac{d^{\alpha}G(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1}\frac{d^nG(\tau)}{d\tau^n}d\tau, \quad t>0\ ,\ n-1<\alpha<n, \tag{1}$$

(14)

The Laplace transform of derivative rule defined in equation (1) is given as

$$L\left\{\frac{d^{\alpha}G(t)}{dt^{\alpha}}\right\} = s^{\alpha}L\{\overline{G}(s)\} - \sum_{k=0}^{n-1} G^{(k)}(0^{+})\,s^{\alpha-1-k}, \quad n-1<\alpha<n. \tag{2}$$

Where

s is transform parameter

n is positive integer.

Governing Equation of Temperature Distribution Function with Boundaries

The heat equation for a rectangular plate in frame of time fractional order derivative $0<\alpha\le 2$ with certain boundaries is as

$$\frac{\partial}{\partial x}\left(k(x,\theta)\,\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,\theta)\frac{\partial\theta}{\partial y}\right) = \rho\, C(x,\theta)\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}} \tag{3}$$

[18]

$$\left.\left(k(x,\theta)\,\frac{\partial\theta}{\partial x} - \varepsilon_1(\theta-\theta_0)\right)\right|_{x=0} = Q_1\,\delta\left(y-y_0\right)\delta(t) \tag{4}$$

$$\left.\left(k(x,\theta)\,\frac{\partial\theta}{\partial x} + \varepsilon_2(\theta-\theta_0)\right)\right|_{x=a} = Q_1\,\delta\left(y-y_0\right)\delta(t) \tag{5}$$

$$\left.\left(k(x,\theta)\,\frac{\partial\theta}{\partial y} - \varepsilon_1(\theta-\theta_0)\right)\right|_{y=0} = 0 \tag{6}$$

$$\left.\left(k(x,\theta)\,\frac{\partial\theta}{\partial y} + \varepsilon_2(\theta-\theta_0)\right)\right|_{y=b} = 0 \tag{7}$$

$$\theta=\theta_0\ ,\ \ at\ \ t=0\ ,\quad 0<\alpha\le 2 \tag{8}$$

[4]

$$\frac{\partial\theta}{\partial t} = 0\ ,\ at\ \ t=0\ \ ,\ \ 1<\alpha\le 2 \tag{9}$$ [4]

In above equations, $\theta(x,y,t)$ is stands for distribution of temperature at any time t, θ_0 is the distribution of temperature in surrounding medium, further a impulsive heat with Q_1 strength is defined as $Q_1\,\delta(y-y_0)\delta(t)$ at the point $(x=0, y=y_0)$ and $(x=a, y=y_0)$, next $k(x,\theta)$ is thermal conductivity and $c(x,\theta)$ is specific heat capacity of the material plate, furthermore ε_1, ε_2 refers here for the coefficients of heat transfer.

Displacement expression for thermal stresses

The rectangular coordinate system under no body forces relation between strain displacement components as:

$$e_{xx} = \frac{\partial u_x}{\partial x},\ e_{yy} = \frac{\partial u_y}{\partial y},$$
$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \quad (10)$$

Equations of equilibrium written as in [26]

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \quad (11)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (12)$$

Here, the component of displacement along plane x-axis is u_x and along plane y-axis is u_y.

Next, Equation (13) to equation (15) shows the plane stress and Equation (16) to equation (18) representing the plane strain

$$\sigma_{xx} = \frac{2G(x,\theta)}{1-v(x,\theta)} \begin{bmatrix} e_{xx} + v(x,\theta)e_{yy} - \xi(x,\theta)\theta \\ - v(x,\theta)\xi(x,\theta)\theta \end{bmatrix} \quad (13)\ [7]$$

$$\sigma_{yy} = \frac{2G(x,\theta)}{1-v(x,\theta)} \begin{bmatrix} v(x,\theta)e_{xx} + e_{yy} - \xi(x,\theta)\theta \\ - v(x,\theta)\xi(x,\theta)\theta \end{bmatrix} \quad (14)\ [7]$$

$$\sigma_{xy} = 2G(x,\theta)e_{xy} \quad (15)\ [7]$$

$$\sigma_{xx} = \frac{2G(x,\theta)}{1-2v(x,\theta)} \begin{bmatrix} e_{xx} - v(x,\theta)e_{xx} + v(x,\theta)e_{yy} \\ - \xi(x,\theta)\theta - v(x,\theta)\xi(x,\theta)\theta \end{bmatrix} \quad (16)[7]$$

$$\sigma_{yy} = \frac{2G(x,\theta)}{1-2v(x,\theta)} \begin{bmatrix} v(x,\theta)e_{xx} + e_{yy} - v(x,\theta)e_{yy} \\ - \xi(x,\theta)\theta - v(x,\theta)\xi(x,\theta)\theta \end{bmatrix} \quad (17)$$

$$\sigma_{xy} = 2G(x,\theta)e_{xy} \quad (18)$$

Here, $G(x,\theta)$ stands for shear elasticity modulus, $a(x,\theta)$ refers for thermal linear expansion coefficient and $v(x,\theta)$ is Poisson's ratio.

A plane stress field equation (19) is obtained by placing the equation (10) and (13) to (15) in (11) and (12);

$$\frac{2G(x,\theta)}{1-v(x,\theta)} \times \left\{ \frac{\partial^2 u_x}{\partial x^2} + v(x,\theta)\frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial u_y}{\partial y}\frac{\partial v(x,\theta)}{\partial x} - \frac{\partial}{\partial x}(\xi(x,\theta)\theta) - v(x,\theta)\frac{\partial}{\partial x}(\xi(x,\theta)\theta) - (\xi(x,\theta)\theta)\frac{\partial v(x,\theta)}{\partial x} \right\} + \frac{\partial}{\partial x}\frac{2G(x,\theta)}{1-v(x,\theta)} \times \left\{ \frac{\partial u_x}{\partial x} + v(x,\theta)\frac{\partial u_y}{\partial y} - (\xi(x,\theta)\theta) - v(x,\theta)(\xi(x,\theta)\theta) \right\}$$

$$\frac{\partial}{\partial y}G(x,\theta)\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + G(x,\theta)\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) = 0$$

$$\frac{2G(x,\theta)}{1-v(x,\theta)} \left\{ \frac{\partial^2 u_y}{\partial y^2} + v(x,\theta)\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial u_x}{\partial x}\frac{\partial v(x,\theta)}{\partial y} - v(x,\theta)\frac{\partial}{\partial y}(\xi(x,\theta)\theta) - \frac{\partial}{\partial y}(\xi(x,\theta)\theta) \right.$$

$$-\left(\xi(x,\theta)\theta\right)\frac{\partial v(x,\theta)}{\partial y}\Bigg\}$$

$$+\frac{\partial}{\partial y}\frac{2G(x,\theta)}{1-v(x,\theta)}\left\{\begin{array}{l}\frac{\partial u_y}{\partial y}+v(x,\theta)\frac{\partial u_x}{\partial x}\\ -\left(\xi(x,\theta)\theta\right)\\ -v(x,\theta)\left(\xi(x,\theta)\theta\right)\end{array}\right\}$$

$$\frac{\partial}{\partial x}G(x,\theta)\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)$$

$$+G(x,\theta)\left(\frac{\partial^2 u_x}{\partial x\partial y}+\frac{\partial^2 u_y}{\partial x^2}\right)=0 \qquad (19)$$

[10, 12, 33]

Similarly, the plane strain field equation (20) obtained by using (10) and (16) to (18) in (11) and (12);

$$\frac{2G(x,\theta)}{1-2v(x,\theta)}\left\{\begin{array}{l}\frac{\partial^2 u_x}{\partial x^2}-v(x,\theta)\frac{\partial^2 u_x}{\partial x^2}\\ +v(x,\theta)\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial u_y}{\partial y}\frac{\partial v(x,\theta)}{\partial x}\\ -\frac{\partial}{\partial x}\left(\xi(x,\theta)\theta\right)\end{array}\right.$$

$$\left.\begin{array}{l}-v(x,\theta)\frac{\partial}{\partial x}\left(\xi(x,\theta)\theta\right)\\ -\left(\xi(x,\theta)\theta\right)\frac{\partial v(x,\theta)}{\partial x}\end{array}\right\}$$

$$+\frac{\partial}{\partial x}\frac{2G(x,\theta)}{1-2v(x,\theta)}\left\{\begin{array}{l}\frac{\partial u_x}{\partial x}+v(x,\theta)\frac{\partial u_y}{\partial y}\\ -\left(\xi(x,\theta)\theta\right)\\ -v(x,\theta)\left(\xi(x,\theta)\theta\right)\end{array}\right\}+$$

$$\frac{\partial}{\partial y}G(x,\theta)\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)$$

$$+G(x,\theta)\left(\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_y}{\partial x\partial y}\right)=0$$

$$\frac{2G(x,\theta)}{1-2v(x,\theta)}\left\{\begin{array}{l}\frac{\partial^2 u_y}{\partial y^2}-v(x,\theta)\frac{\partial^2 u_y}{\partial y^2}\\ +v(x,\theta)\frac{\partial^2 u_x}{\partial x\partial y}\\ +\frac{\partial u_x}{\partial x}\frac{\partial v(x,\theta)}{\partial y}\\ -\frac{\partial}{\partial y}\left(\xi(x,\theta)\theta\right)\end{array}\right.$$

$$\left.\begin{array}{l}-v(x,\theta)\frac{\partial}{\partial y}\left(\xi(x,\theta)\theta\right)\\ -\left(\xi(x,\theta)\theta\right)\frac{\partial v(x,\theta)}{\partial y}\end{array}\right\}$$

$$+\frac{\partial}{\partial y}\frac{2G(x,\theta)}{1-2v(x,\theta)}\times$$

$$\left\{\begin{array}{l}\frac{\partial u_y}{\partial y}+v(x,\theta)\frac{\partial u_x}{\partial x}-\left(\xi(x,\theta)\theta\right)\\ -v(x,\theta)\left(\xi(x,\theta)\theta\right)\end{array}\right\}$$

$$\frac{\partial}{\partial x}G(x,\theta)\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)$$

$$+G(x,\theta)\left(\frac{\partial^2 u_x}{\partial x\partial y}+\frac{\partial^2 u_y}{\partial x^2}\right)=0 \qquad (20)$$

[1, 7, 10, 35, 36]

Next, φ is Goodier's displacement function, Õ and ψ are the Boussinesq Harmonic functions used to express the solutions of (7) and (8) as

$$u_x=\frac{\partial\phi}{\partial x}+\frac{\partial\varphi}{\partial x}+2\frac{\partial\psi}{\partial y} \qquad (21)[2]$$

$$u_y=\frac{\partial\phi}{\partial y}+\frac{\partial\varphi}{\partial y}-2\frac{\partial\psi}{\partial x} \qquad (22)[2]$$

Further, φ, Õ and ψ satisfies the following conditions

$$\left.\begin{aligned}&\nabla^2\phi = K(x,\theta)(\theta-\theta_0)\\&\nabla^2\phi = 0\\&\nabla^2\psi = 0\end{aligned}\right\} \quad (23)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \text{ and}$$

$$K(x,\theta)=\frac{1+v(x,\theta)}{1-v(x,\theta)}\xi(x,\theta)$$

On substituting (21) and (22) in (13) to (19), the corresponding resultant stress function in plane stress are obtained as

$$\sigma_{xx} = \frac{2G(x,\theta)}{1-v(x,\theta)}\left\{\begin{aligned}&\frac{\partial^2\phi}{\partial x^2}\\&+v(x,\theta)\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\varphi}{\partial x^2}\\&+v(x,\theta)\frac{\partial^2\varphi}{\partial y^2}\\&+2\frac{\partial^2\psi}{\partial x\partial y}\end{aligned}\right.$$

$$\left.\begin{aligned}&-2v(x,\theta)\frac{\partial^2\psi}{\partial x\partial y}\\&-\xi(x,\theta)\theta-v(x,\theta)\xi(x,\theta)\theta\end{aligned}\right\} \quad (24)[1,2,10]$$

$$\sigma_{yy} = \frac{2G(x,\theta)}{1-v(x,\theta)}\times$$

$$\left\{\begin{aligned}&v(x,\theta)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\varphi}{\partial y^2}\\&+v(x,\theta)\frac{\partial^2\varphi}{\partial x^2}-2\frac{\partial^2\psi}{\partial x\partial y}\end{aligned}\right.$$

$$\left.\begin{aligned}&+2v(x,\theta)\frac{\partial^2\psi}{\partial x\partial y}\\&-\xi(x,\theta)\theta-v(x,\theta)\xi(x,\theta)\theta\end{aligned}\right\} \quad (25)$$

$$\sigma_{xy} = 2G(x,\theta)\left\{\begin{aligned}&\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial x\partial y}\\&-\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\end{aligned}\right\} \quad (26)[1, 25]$$

On substituting (21) and (22) in (16) to (20), the corresponding resultant stress function in plane strain are obtained as

$$\sigma_{xx} = \frac{2G(x,\theta)}{1-2v(x,\theta)}\left\{\begin{aligned}&\frac{\partial^2\phi}{\partial x^2}\\&-v(x,\theta)\frac{\partial^2\phi}{\partial x^2}\\&+v(x,\theta)\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\varphi}{\partial x^2}\\&-v(x,\theta)\frac{\partial^2\varphi}{\partial x^2}\\&+v(x,\theta)\frac{\partial^2\varphi}{\partial y^2}\end{aligned}\right.$$

$$\left.\begin{aligned}&+2\frac{\partial^2\psi}{\partial x\partial y}-4v(x,\theta)\frac{\partial^2\psi}{\partial x\partial y}\\&-\xi(x,\theta)\theta-v(x,\theta)\xi(x,\theta)\theta\end{aligned}\right\} \quad (27)$$

$$\sigma_{yy} = \frac{2G(x,\theta)}{1-2v(x,\theta)}\left\{\begin{aligned}&v(x,\theta)\frac{\partial^2\phi}{\partial x^2}\\&+\frac{\partial^2\phi}{\partial y^2}-v(x,\theta)\frac{\partial^2\phi}{\partial y^2}\\&+v(x,\theta)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\\&-v(x,\theta)\frac{\partial^2\varphi}{\partial y^2}\end{aligned}\right.$$

$$\left.\begin{aligned}&-2\frac{\partial^2\psi}{\partial x\partial y}+4v(x,\theta)\frac{\partial^2\psi}{\partial x\partial y}\\&-\xi(x,\theta)\theta-v(x,\theta)\xi(x,\theta)\theta\end{aligned}\right\} \quad (28)$$

$$\sigma_{xy} = 2G(x,\theta)\left\{ \begin{array}{l} \dfrac{\partial^2 \phi}{\partial x \partial y} + \dfrac{\partial^2 \varphi}{\partial x \partial y} \\ -\dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2} \end{array} \right\} \quad (29)[1]$$

The boundaries for the condition of traction-free surface is

$$\sigma_{xx} = 0; \; at \quad x = 0 \;\&\; x = a \quad (30)[2]$$

$$\sigma_{yy} = 0; \; at \quad y = 0 \;\&\; y = a \quad (31)[2]$$

SOLUTIONS

Heat conduction problem

The integral transform and its inverse is used to examine the heat transfer with boundary conditions $0 < x \le a$ for the function $f(x,y,t)$ defined using equation [27] as

$$\overline{f}(\beta_n, y, t) = \int_{x'=0}^{a} S(\beta_n, x')\, f(x', y, t)\, dx' \quad (32)[1]$$

$$f(x,y,t) = \sum_{n=1}^{\infty} S(\beta_n, x)\, \overline{f}(\beta_n, y, t) \quad (33)$$

the kernel of the transformation is

$$S(\beta_n, x) = L_1\left(\beta_n \cos \beta_n x + \varepsilon_1 \sin \beta_n x\right) \quad (34)$$

here

$$L_1 = \sqrt{\frac{2(\beta_n^2 + \varepsilon_2^2)}{(\beta_n^2 + \varepsilon_1^2)\left(a(\beta_n^2 + \varepsilon_2^2) + \varepsilon_2\right) + \varepsilon_1(\beta_n^2 + \varepsilon_2^2)}} \quad (35)$$

Where, the non-negative roots of the transcendental equation is β_n;

$$\tan \beta_n a = \frac{\beta_n(\varepsilon_1 + \varepsilon_2)}{\beta_n^2 - \varepsilon_1 \varepsilon_2} \quad (36)$$

The integral transform and its inversion formula with respect to $0 < y \le b$ for the function $\overline{f}(\beta_n, y, t)$ defined by following equation [27] as;

$$\overline{\overline{f}}(\beta_n, \beta_m, t) = \int_{y'=0}^{b} S(\beta_m, y')\, \overline{f}(\beta_n, y', t)\, dy' \quad (37)$$

$$\overline{f}(\beta_n, y, t) = \sum_{m=1}^{\infty} S(\beta_m, y)\, \overline{\overline{f}}(\beta_n, \beta_m, t) \quad (38)$$

the kernel of the transformation is

$$S(\beta_m, y) = L_2\left(\beta_m \cos \beta_m y + \varepsilon_1 \sin \beta_m y\right) \quad (39)$$

here

$$L_2 = \sqrt{\frac{2(\beta_m^2 + \varepsilon_2^2)}{(\beta_m^2 + \varepsilon_1^2)\left(a(\beta_m^2 + \varepsilon_2^2) + \varepsilon_2\right) + \varepsilon_1(\beta_m^2 + \varepsilon_2^2)}} \quad (40)$$

Where here, β_m denotes the non-negative roots of the transcendental equation

$$\tan \beta_m b = \frac{\beta_m(\varepsilon_1 + \varepsilon_2)}{\beta_m^2 - \varepsilon_1 \varepsilon_2} \quad (41)$$

On using Popovych et al. [2, 4, 9, 10, 13, 3] the Kirchhoff's variable is introduced as

$$T(\Theta) = \int_{\theta_0}^{\theta} k(x,\theta)\, d\theta \quad (42)$$

By using (42), equation (3) to (9) can be rewritten as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\kappa} \frac{\partial^\alpha T}{\partial t^\alpha} \quad (43)$$

$$\left. \frac{\partial T}{\partial x} - \varepsilon_1 T \right|_{x=0} = Q_1\, \delta\left(y - y_0\right) \delta(t) \quad (44)$$

$$\left. \frac{\partial T}{\partial x} + \varepsilon_2 T \right|_{x=a} = Q_1\, \delta\left(y - y_0\right) \delta(t) \quad (45)$$

$$\left. \frac{\partial T}{\partial y} - \varepsilon_1 T \right|_{y=0} = 0 \quad (46)$$

$$\left. \frac{\partial T}{\partial y} + \varepsilon_2 T \right|_{y=b} = 0 \quad (47)$$

$$T = T_0, \;\; at \;\; t = 0, \;\; 0 < \alpha \le 2 \quad (48)$$

$$\frac{\partial T}{\partial t} = 0, \;\; at \;\; t = 0, \;\; 1 < \alpha \le 2 \quad (49)[30]$$

Where $\kappa = k_0 / C_0 \rho_0$, k_0 is reference thermal conductive, C_0 calorific capacity, and ρ_0 is density of material.

Employing Laplace transform to the equation (43) and using initial boundaries (48)-(49), we obtain

$$\frac{\partial^2 T^*}{\partial x^2} + \frac{\partial^2 T^*}{\partial y^2} = \frac{1}{\kappa} s^\alpha T^* \quad (50)$$

$$\left. \frac{\partial T^*}{\partial x} - \varepsilon_1 T^* \right|_{x=0} = Q_1 \delta(y - y_0) \quad (51)$$

$$\left. \frac{\partial T^*}{\partial x} + \varepsilon_2 T^* \right|_{x=a} = Q_1 \delta(y - y_0) \quad (52)$$

$$\left. \frac{\partial T^*}{\partial y} - \varepsilon_1 T^* \right|_{y=0} = 0 \quad (53)$$

$$\left. \frac{\partial T^*}{\partial y} + \varepsilon_2 T^* \right|_{y=b} = 0 \quad (54)$$

Employing the integral transform defined in equation (32) on (50) with respect to variable x;

$$-\beta_n^2 \overline{T}^* + \left[L_1 \beta_n + L_1(\beta_n \cos\beta_n a + \varepsilon_1 \sin\beta_n a)\right] \times Q_1 \delta(y - y_0) + \frac{\partial \overline{T}^*}{\partial y} = \frac{1}{\kappa} s^\alpha \overline{T}^* \quad (55)$$

$$\left. \frac{\partial \overline{T}^*}{\partial y} - \varepsilon_1 \overline{T}^* \right|_{y=0} = 0 \quad (56)$$

$$\left. \frac{\partial \overline{T}^*}{\partial y} + \varepsilon_2 \overline{T}^* \right|_{y=b} = 0 \quad (57)$$

Next, applying the transformation defined in (37) to (46) w. r. t. variable y;

$$L_2 \left[L_1 \beta_n + L_1(\beta_n \cos\beta_n a + \varepsilon_1 \sin\beta_n a)\right] \times Q_1 y_0 \kappa(\beta_m \cos\beta_m y_0 + \varepsilon_1 \sin\beta_m y_0) = \left\{\kappa(\beta_n^2 + \beta_m^2) + s^\alpha\right\} \overline{\overline{T}}^* \quad (58)$$

Further, using inversion of Laplace transform to (58) we get

$$\overline{\overline{T}} = t^{\alpha-1} E_\alpha\left(-\kappa(\beta_n^2 + \beta_m^2) t^\alpha\right) \times L_2 \left[L_1 \beta_n + L_1(\beta_n \cos\beta_n a + \varepsilon_1 \sin\beta_n a)\right] \times Q_1 y_0 \kappa(\beta_m \cos\beta_m y_0 + \varepsilon_1 \sin\beta_m y_0) \quad (59)$$

Where, $E_\alpha(.)$ represents the Mittag-Leffler-type function.

Next, using the inversion formula defined in Eq. (38) on (50), we obtain

$$\overline{T} = \sum_{m=1}^{\infty} \left\{ S(\beta_m, y) t^{\alpha-1} E_\alpha\left(-\kappa(\beta_n^2 + \beta_m^2) t^\alpha\right) \times L_2 \left[L_1 \beta_n + L_1(\beta_n \cos\beta_n a + \varepsilon_1 \sin\beta_n a)\right] \times Q_1 y_0 \kappa(\beta_m \cos\beta_m y_0 + \varepsilon_1 \sin\beta_m y_0) \right\} \quad (60)$$

Finally, employing inversion formula defined in (33) to (60), we obtain expression for temperature distribution as

$$T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ S(\beta_n, x) S(\beta_m, y) t^{\alpha-1} E_\alpha \times \left(-\kappa(\beta_n^2 + \beta_m^2) t^\alpha\right) L_2 \times \left[L_1 \beta_n + L_1(\beta_n \cos\beta_n a + \varepsilon_1 \sin\beta_n a)\right] \times Q_1 y_0 \kappa(\beta_m \cos\beta_m y_0 + \varepsilon_1 \sin\beta_m y_0) \right\} \quad (61)$$

Next, by substituting the inverse variable transformation from θ to T as in [6], the required expression for temperature distribution is

$$\theta(x, y, t) \cong \theta_0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\begin{array}{l} \{1/u(x) \exp(\varpi_1 \theta_0)\} \times \\ S(\beta_n, x) S(\beta_m, y) t^{\alpha-1} E_\alpha \times \\ \left(-\kappa(\beta_n^2 + \beta_m^2) t^\alpha\right) \\ \times L_2 \left[L_1 \beta_n + L_1(\beta_n \cos\beta_n a + \varepsilon_1 \sin\beta_n a)\right] \times \\ Q_1 y_0 \kappa(\beta_m \cos\beta_m y_0 + \varepsilon_1 \sin\beta_m y_0) \end{array} \right] \quad (62)$$

Where

$$u(x) = \left[f_m(x)(k_{mo} - k_{co}) + k_{co}\right]$$

$f_m(x) = 1 - x^\gamma$ for $\gamma \geq 0$

Thermoelastic Equations

The solution of thermoelastic Goodier potential displacement function obtained by using equation (53) in (23)

$$\phi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\begin{array}{l}\left\{\begin{array}{l}K(x,T)g_1(x,y)/\\ \left[g_2(x,y)(u(x))^3\exp(\varpi_1\theta_0)\right]\end{array}\right\}\times\\ S(\beta_n,x)S(\beta_m,y)t^{\alpha-1}E_\alpha\times\\ \left(-\kappa\left(\beta_n^2+\beta_m^2\right)t^\alpha\right)\end{array}\right.$$

$$\left.\begin{array}{l}\times L_2\left[L_1\beta_n + L_1(\beta_n\cos\beta_n a+\varepsilon_1\sin\beta_n a)\right]\times\\ Q_1 y_0\kappa(\beta_m\cos\beta_m y_0+\varepsilon_1\sin\beta_m y_0)\end{array}\right] \quad (63)$$

where

$$g_1(x,y)=S(\beta_n,x)\times S(\beta_m,y)$$

$$g_2(x,y)=S(\beta_m,y)\times [u(x)g_3'(x)-2g_3(x)u'(x)] + S(\beta_n,x)S''(\beta_m,y)(u(x))^2$$

$$g_3(x)=u(x)S'(\beta_n,x)-u'(x)S(\beta_n,x)$$

Expressions for Boussinesq harmonic functions Õ and ø satisfying equation (23) is assumed as

$$\varphi=\psi=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sinh(p_1 t)\left[\begin{array}{l}B_n\sin(\beta_n x)\\ +D_n\cos(\beta_n x)\end{array}\right] \times(\sin(\beta_m y)+\cos(\beta_m y)) \quad (64)$$

Where B_n, D_n are constants.

Using the values of φ, ϕ, ψ from (63) and (64) in (21) to (22), we obtain

$$u_x=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\partial\phi}{\partial x}+\sinh(p_1 t)\beta_n\times [B_n\cos(\beta_n x)-D_n\sin(\beta_n x)]\times (\sin(\beta_m y)+\cos(\beta_m y)) + 2\beta_m\sinh(p_1 t)\left[\begin{array}{l}B_n\cos(\beta_n x)\\ +D_n\sin(\beta_n x)\end{array}\right]\times (\cos(\beta_m y)-\sin(\beta_m y))\right. \quad (65)$$

$$u_y=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\partial\phi}{\partial y}+\sinh(p_1 t)\beta_m\times [B_n\sin(\beta_n x)+D_n\cos(\beta_n x)]\times (\cos(\beta_m y)-\sin(\beta_m y)) -2\beta_n\sinh(p_1 t)\left[\begin{array}{l}B_n\cos(\beta_n x)\\ -D_n\sin(\beta_n x)\end{array}\right]\times (\cos(\beta_m y)+\sin(\beta_m y))\right. \quad (66)$$

Here, G is SME-shear modulus elasticity, ξ is CLTE-linear thermal coefficient, and v is PR- Poisson ratio in case Functionally Graded Materials dependent on x are expressed as SME, CLTE, and PR of material as G_m, ξ_m, v_m and case of ceramics G_c, ξ_c, v_c with the volume-fractions of material $f_m(x)$ and ceramics, $1-f_m(x)$ as [6]

$$G(x,T)=G_m(T)f_m(x)+G_c(T)\times(1-f_m(x)) \quad (67)$$

$$\xi(x,T)=\xi_m(T)f_m(x)+\xi_c(T)\times(1-f_m(x)) \quad (68)$$

$$v(x,T)=v_m(T)f_m(x)+v_c(T)\times(1-f_m(x)) \quad (69)[6]$$

By using Noda [1], we gave following exponential law:

$$G_j(T)=G_{j0}\exp(\varpi_1 t)$$

$$\xi_j(T)=\xi_{j0}\exp(\varpi_2 t)$$

$$v_j(T)=v_{j0}\exp(\varpi_2 t)$$

$$j=m\ c;\ \varpi_1\le 0\ \varpi_2\ge 0 \quad (70)$$

Here, G_{j0}, a_{j0}, and v_{j0} are the reference values of SME, CLTE, and PR, respectively.

The Equations (63) , (64), (67) to (70) in (24) to (29) gives the plane stress and strain field along the rectangular plate. The values of corresponding stresses B_n and D_n may be found by using the traction-free conditions given by Eq. (30) and (31) in the stresses (24) to (29). Here the resultant stresses and the constants B_n and D_n are found using computer simulation.

Numerical Calculations

For the purpose of numerical analysis alumina is set as the ceramic and nickel as metal to form functionally graded metal-ceramic base.

Where non-dimensional variables following [15] are as:

$$\theta^* = \frac{\theta}{\theta_0}, \quad x^* = \frac{x}{b},$$

$$y^* = \frac{y}{b}, \quad t^* = \frac{\kappa t}{b^2}, \quad u_x^* = \frac{u_x}{K_0 \, \theta_0 \, b},$$

$$u_y^* = \frac{u_y}{K_0 \, \theta_0 \, b},$$

$$\sigma_{xx}^* = \frac{\sigma_{xx}}{E \, G_0 \, \theta_0},$$

$$\sigma_{yy}^* = \frac{\sigma_{yy}}{E \, G_0 \, \theta_0}, \; \sigma_{xy}^* = \frac{\sigma_{xy}}{E \, G_0 \, \theta_0},$$

$$K_0 = \frac{1+v}{1-v} \alpha_0$$

The dimensions used during the numerical calculation are as:

a= 1 cm, b = 2 cm, $T_0 = 320^0 K$

The other related values are as taken:

For Alumina (Ceramic):-

Thermal

conductivity $k = 0.282 \; W / cmK$,

thermal diffusivity

$k_i = 0.083 \times 10^{-6}$ *cm²/s*, Shear modulus $G = 8.8 \times 10^6$ *N/cm²* Coefficient of linear thermal expansion $\alpha = 5.4 \times 10^{-6}/K$, Poisson's ratio $v = 0.23$

For Nickel (Metal):

Thermal

conductivity $k = 0.901 \; W / cmK$,

thermal diffusivity

$k_i = 0.223 \times 10^{-6} \; cm^2 / s$, Shear modulus $G = 7.2 \times 10^6 \; N / cm^2$ Coefficient of linear thermal expansion $\alpha = 14 \times 10^{-6} / K$, Poisson's ratio $v = 0.31$

Analysis of Numerical Results

For numerical computation of dimensionless temperature distribution and thermal stresses for different values of fractional-order parameter $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$, $\alpha = 2$ (depicting weak, normal and strong conductivity) MATEMATICA software is used.

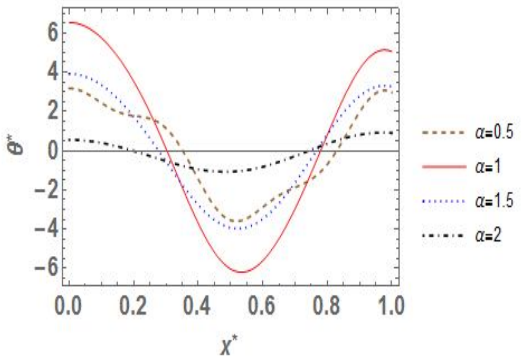

Figure 1(a): Temperature distribution along x axis in homogeneous plate for different fractional parameter

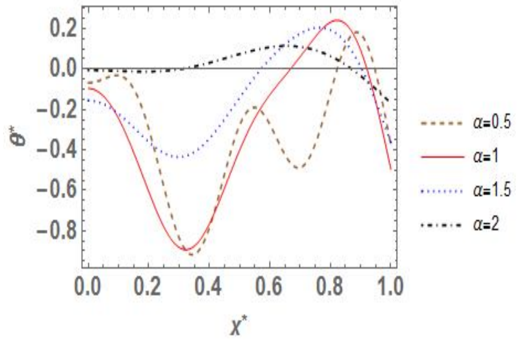

Figure 1(b): Temperature distribution along x axis in nonhomogeneous plate for different fractional parameter

Figure 1(a) and 1(b) represents the variations of θ^* (dimensionless temperature) along x axis in both the homogeneous and non-homogeneous cases for different values of fractional order parameter $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$ and $\alpha = 2$ on fixing $y^* = 0.6$. Variation of temperature is noted positive initially and after it decrease till centre then after it suddenly increases towards outer part for homogeneous case. But in nonhomogeneous plate case distribution initially decreases till $x^* = 0.4$ and then after increases towards outer end. Further speed of propagation of thermal signals are varying directly proportional to the different values of fractional-order parameter α. Furthermore, a non-uniform flow of temperature due to different fractional parameter can be seen clearly in both the cases, Hence distribution with different fractional parameter affects the design of various solid structures which is used in engineering and science.

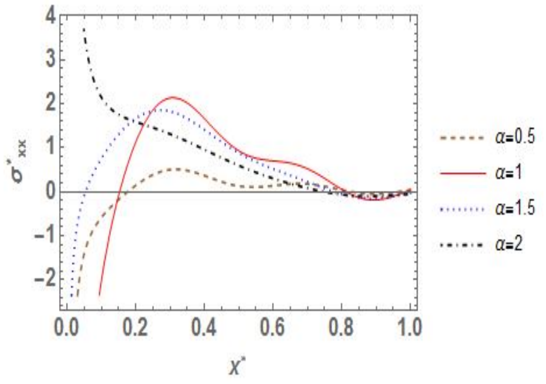

Figure 2(a): Distribution of σ_{xx}^{*} (plane stress field) along x axis in homogeneous plate for different fractional parameters

Figure 2(a), 2(b) and 2(c) represents the variations of dimensionless plane stress field along x axis in the homogeneous case for different values of fractional order parameters $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$ and $\alpha = 2$ on fixing $y^{*} = 0.6$. It is seen that σ_{xx}^{*} goes on increasing towards outer end and reaches at peak near the centre and converges to zero at the end. Stress distribution σ_{yy}^{*} found positive at both the initial and outer end while σ_{xy}^{*} decreases till centre of plate and thereafter it starts increasing towards outer end. Further it is noted that fractional parameters affects the stress variations which may be applicable for design of more stress bearing structures.

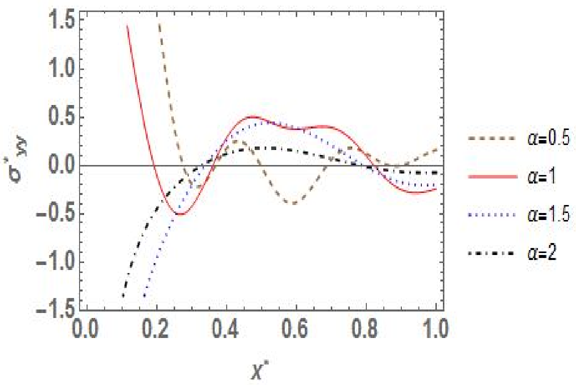

Figure 2(b): Distribution of σ_{yy}^{*} (plane stress field) along x axis in homogeneous plate for different fractional parameter

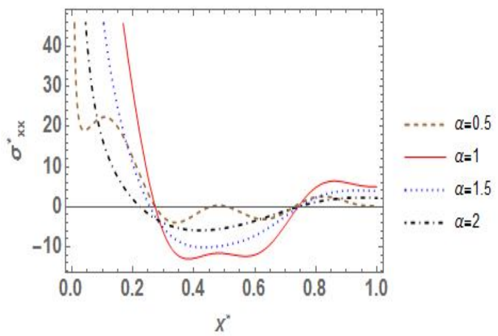

Figure 3(a): Variation of σ_{xx}^{*} (plane stress field) along x axis in nonhomogeneous plate for different fractional parameters

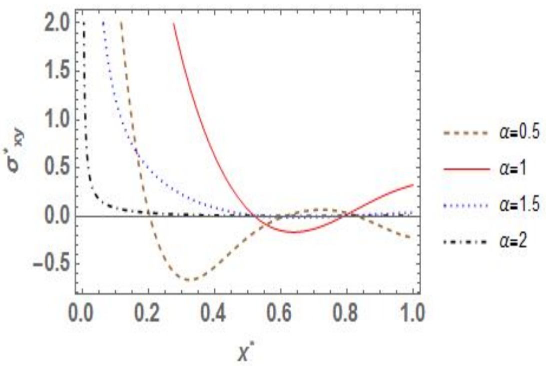

Figure 2(c): Variation of σ_{xy}^{*} (plane stress field) along x axis in homogeneous plate for different fractional parameter

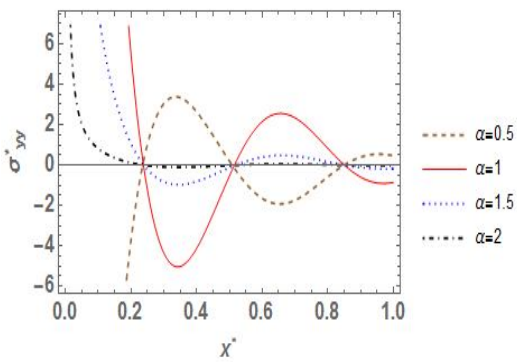

Figure 3(b): Variation of σ_{yy}^{*} (plane stress field) along x axis in nonhomogeneous plate for different fractional parameter

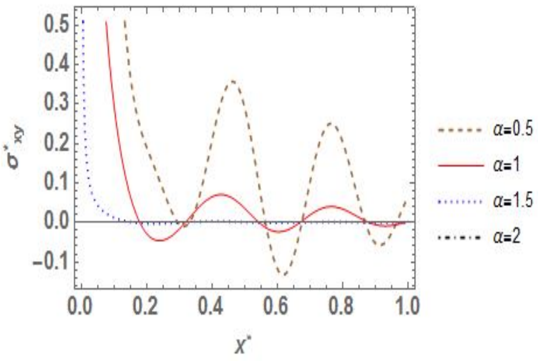

Figure 3(c): Variation of $\sigma_{xy}{}^*$ (plane stress field) along x axis in nonhomogeneous plate for different fractional parameter

Figure 3(a), 3(b) and 3(c) represents the variations of dimensionless plane stress field along x axis in the nonhomogeneous case for different values of fractional order parameters $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$ and $\alpha = 2$ on fixing $y^* = 0.6$. It is seen that initially $\sigma_{xx}{}^*$ goes on decreasing towards outer end and reaches negative near the centre and starts increasing thereafter. Stress distributions $\sigma_{xy}{}^*$ and $\sigma_{yy}{}^*$ found decreasing initially and becomes sinusoidal in nature towards outer end.

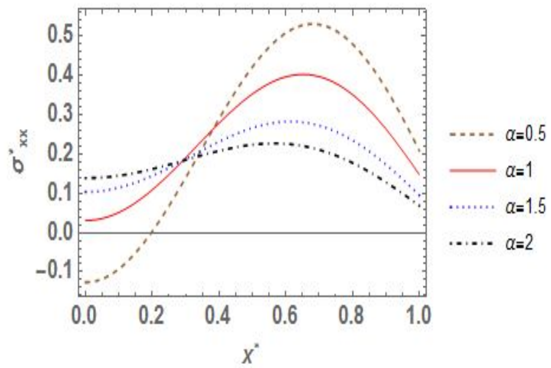

Figure 4(a): Dimensionless stress distribution (plane strain field) along x axis in homogeneous plate for different fractional parameter

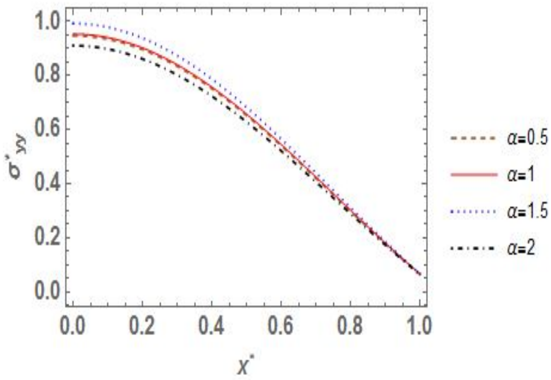

Figure 4(b): Dimensionless stress distribution (plane strain field) along x axis in nonhomogeneous plate for different fractional parameter

Figure 4(a) and 4(b) represents the variations of dimensionless plane strain field along x axis in the homogeneous case for different values of fractional order parameters. It is seen that all the stress components are tensile throughout the plate.

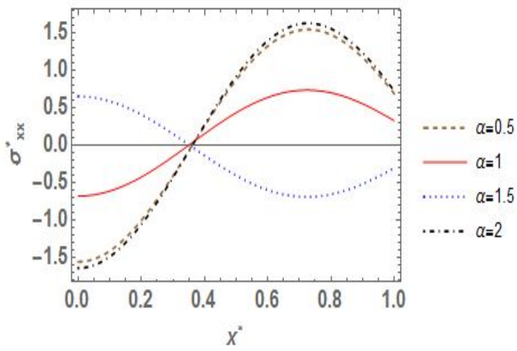

Figure 5(a): Dimensionless stress distribution (plane strain field) along x axis in nonhomogeneous plate for different fractional parameter

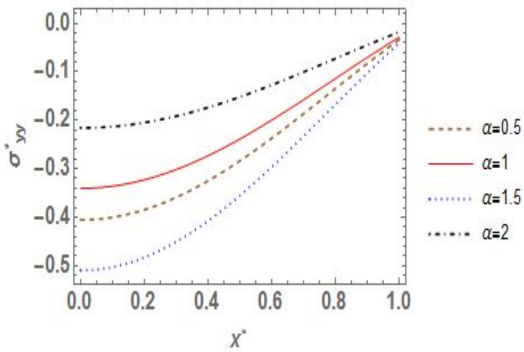

Figure 5(b): Plane strain distribution on rectangular plate(non-homogeneous)

Figure 5(a) and 5(b) represents the variations of dimensionless plane strain field along x axis in the nonhomogeneous case for different values of fractional order parameters. It is seen that σ_{xx}^* is compressive near outer end for some values of fractional parameter while the stress component σ_{yy}^* are tensile over the plate.

Conclusion

Present study adopted the Caputo methodology to generate the expression for analyzing the heat conduction problem related to ceramic based material for temperature and stress distribution along the rectangular plate. The results displays that the temperature distribution is found to be positive at both ends of the plate in homogeneous condition and negative in non-homogeneous conditions. The study indicates that fractional parameter produces more impact on the temperature distribution, plane stress and strain field. The factional parameters also have an effect on the material heat conduction ability.

The present article is very useful for design and construction of new materials and applicable for realistic situations

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