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Certain Properties and Formulae of k-Hypergeometric Functions

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Abstract

Recently the second order homogeneous k-hypergeometric differential equation and the definition of the k- hypergeometric function ${}_2\mathcal{F}_{1,k}[(a,k),(b,k);(c,k);z]$ as its solution at the origin have been presented. In this paper, we expound on the certain properties of the above function. The main object of this paper is to establish certain transformation formulae for the above function. In addition, the various important differential formulae for the k-hypergeometric functions are investigated. Since the result derived here are general in nature, it is expected that they will be useful addition in the theoretical development of the Gauss hypergeometric function.

Keywords: k-pochhammer symbol, k-hypergeometric functions, Hypergeometric transformation.

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Introduction

The Gauss hypergeometric equation which is expressed by

$$z(1-z)\frac{d^2y}{dz^2}+[c-(1+a+b)z]\frac{dy}{dz}-aby=0 \qquad (1.1)$$

with 3 regular singularities {0, 1, ∞} has been extensively studied by various authors including Coddington[4], Campos[5], Gasper[7], RainVille[12], Slater[13], Whittaker.[15]

A hypergeometric series solution ${}_2\mathcal{F}_1$ $[a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$ of (1) can be derived by the Frobenius method is convergent in the region $R=\{z:|z|<1\}, when\ c\notin z, z^{1-c}{}_2F_1[a-c+1,b-c+1;2-c;z]$ is also a linearly independent solution of (1.1) and convergent in the region R.

The k-hypergeometric function has been proposed as a generalization of the Gauss hypergeometric function.

The second order homogeneous k-hypergeometric equation is given by

$$kz(1-kz)\frac{d^2y}{dz^2}+[c-(k+a+b)kz]\frac{dy}{dz}-aby=0 \qquad (1.2)$$

And using Frobenious method, we have

$$y_1(z)={}_2\mathcal{F}_{1,k}[(a,k),(b,k);(c,k);z]=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!}; \qquad c\in N_0$$

where $(\alpha)_{n,k}=\alpha(\alpha+k)(\alpha+2k)\ldots(\alpha+(n-1)k), \alpha\neq 0$ and $(\alpha)_{0,k}=1$, $k>0$ is a solution of (1.2). This is the k-hypergeometric function. We can show that $z^{1-\frac{c}{k}}y_1(z)$ is again a solution of (1.2) but with different coefficients.

Using this we can prove that

$$y_2(z)=z^{1-\frac{c}{k}}{}_2\mathcal{F}_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z]$$

is second solution for $c-2k\notin N_0$. This gives two linearly independent solutions of (1.2) $c\notin Z.$

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Here one complete solution of (2) is

$$y(z)=A\,{}_2\mathcal{F}_{1,k}[(a,k),(b,k);(c,k);z] + Bz^{1-\frac{c}{k}}{}_2\mathcal{F}_{1,k}[(a+k-c,k),(b+k-c,k);(2k-c,k);z] \qquad (1.3)$$

for $|z|<\frac{1}{k}$ $c\notin Z$ where A and B are constant. One may refer to the works of Abdalla[1], Ahmad[2], Ali[3], Krasniqi[8], Mubeen [9-11] and Shengfeng.[14]

The study of transformation formulae for the Gauss hypergeometric function have occupied the attention of many authors. Here in the present paper by substitution and using the complete solution discussed earlier, we have deduce the transformation relation and the generalised form of Euler identity for the k- hypergeometric function. Further we have derived some important differential formulae for the above function.

Preliminaries

In this section, we briefly review some basic definitions and facts concerning the Fuchsion type differential equation and k-hypergeometric series. Some surveys and literature for k-hypergeometric series and the k- hypergeometric differential equation can be found in Diaz et al.,[6] Abdalla,[1] Ahmad,[2] Ali,[3] Krasniqi,[8] Mubeen[9-11] and Shengfeng.[14]

Definition 2.1: Assume that p(z) and q(z) be two complex valued functions. Let a second order differential equation in standard form:

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y(z) = 0 \quad (2.1)$$

Then the method about finding an infinite series solution of equation (2.1) is called the Frobenius method.

Definition 2.2: If $z = z_0$ is a singularity of (2.1), then $z = z_0$ is a regular singularity of (2.1) if and only if $(z - z_0)p(z), (z - z_0)^2 q(z)$ are analytic in $\{z: |z - z_0| < R\}$, where R is a positive real number.

Definition 2.3: The Pochhammer k-symbol $(\alpha)_{n,k}$ is defined by:

$$(\alpha)_{n,k} = \alpha(\alpha + k)(\alpha + 2k) \ldots (\alpha + (n-1)k), \alpha \neq 0 \quad (2.2)$$

$(\alpha)_{0,k} = 1$ where $k > 0$

Definition 2.4: The k-hypergeometric series with three parameters a, b, and c is defined as:

$${}_2\mathcal{F}_{1,k}\,[(\mathrm{a,k}), (b,k); (c,k); z] = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} \quad (2.3)$$

Transformation Formulae of the k-hypergeometric Function

In this section we have derived the transformation formulae associated to the k-hypergeometric function in the form of following theorems:

Theorem 3.1: The following the transformation formula for k-hypergeometric function holds:

$${}_2\mathcal{F}_{1,k}[(a,k),(b,k);(c,k);z] = \{(1-kz)\}^{-\frac{a}{k}}{}_2\mathcal{F}_{1,k}\left[(\mathrm{a,k}),(c-b,k);(c,k);\frac{-z}{1-kz}\right] \quad (3.1)$$

Proof: Let

$$y = \{k(1-kz)\}^{-\frac{a}{k}} w$$

$$\frac{dy}{dz} = \{k(1-kz)\}^{-\frac{a}{k}}\frac{dw}{dz} + ak(k-k^2z)^{-\frac{a}{k}-1}w$$

$$\frac{d^2y}{dz^2} = \{k(1-kz)\}^{-\frac{a}{k}}\frac{d^2w}{dz^2} + 2ak(k-k^2z)^{-\frac{a}{k}-1}\frac{dw}{dz} + ak(a+k)k(k-k^2z)^{-\frac{a}{k}-2}w$$

On plungging everything into the k-analogues hypergeometric differential equation (1.2) we get,

$$kz(1-kz)\left[\frac{d^2w}{dz^2} + 2ak(k-k^2z)^{-1}\frac{dw}{dz} + a(a+k)k^2(k-k^2z)^{-2}w\right] + \{c - (k+a+b)kz\}\left[\frac{dw}{dz} + ak(k-k^2z)^{-1}w\right] - abw = 0$$

$$kz(1-kz)^2\frac{d^2w}{dz^2} + (1-kz)[c + (a-b-k)kz]\frac{dw}{dz} + a(c-b)w = 0 \quad (3.2)$$

Now, let

$$t = \frac{-z}{(1-kz)}, \textit{and then } kz = \frac{-kt}{(1-kt)}, 1-kz = \frac{1}{(1-kt)}$$

$$\frac{dw}{dz} = -(1-kt)^2\frac{dw}{dt}$$

$$\frac{d^2w}{dz^2} = \left[(1-kt)^4\frac{d^2w}{dt^2} - 2k(1-kt)^3\frac{dw}{dt}\right]$$

Equation (3.2) becomes

$$\left(\frac{-kt}{(1-kt)}\right)\left(\frac{1}{(1-kt)}\right)^2\left[(1-kt)^4\frac{d^2w}{dt^2} - 2k(1-kt)^3\frac{dw}{dt}\right] + \left(\frac{1}{(1-kt)}\right)[c + (a-b-k)]\left(\frac{-kt}{(1-kt)}\right)\left(-(1-kt)^2\frac{dw}{dt}\right) + a(c-b)w = 0$$

$$kt(1-kt)\frac{d^2w}{dt^2} + [c - (k+a+c-b)kt]\frac{dw}{dt} - a(c-b)w = 0 \quad (3.3)$$

Which is a k-hypergeometric type differential equation with different coefficient and whose solution can be written as

$$w = {}_2\mathcal{F}_{1,k}[(a,k),(c-b,k);(c,k);t]$$

$$y = \{k(1-kz)\}^{-\frac{a}{k}}{}_2\mathcal{F}_{1,k}\left[(\mathrm{a,k}),(c-b,k);(c,k);\frac{-z}{1-kz}\right]$$

Now we have two linearly independent solution of equation (1.2) as

$$y_1(z) = {}_2\mathcal{F}_{1,k}[(a,k),(b,k)(c,k);z]$$

$$y_2(z) = z^{\left(1-\frac{c}{k}\right)}\,{}_2\mathcal{F}_{1,k}[(k+a-c,k),(k+b-c,k);(2k-c,k);z]$$

So, as usual we have $y = Ay_1 + By_2$

And by letting $k = 1 \textit{ and } z \to 0$, we conclude $B = 0 \textit{ and } A = (k)^{-\frac{a}{k}}, \textit{so that if}$

$$|z| < \frac{1}{k} \textit{ and } |1-kz| < 1,$$

$${}_2\mathcal{F}_{1,k}[(a,k),(b,k);(c,k);z] = \{(1-kz)\}^{-\frac{a}{k}}{}_2\mathcal{F}_{1,k}\left[(\mathrm{a,k}),(c-b,k);(c,k);\frac{-z}{1-kz}\right]$$

Theorem 3.2: If $|z| < \frac{1}{k}$ *and* $\left|\frac{z}{1-kz}\right| < \frac{1}{k}$, then the following transformation formula for k-hypergeometric function holds:

$${}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z] = (1-\mathrm{kz})^{\frac{1}{k}(c-a-b)}{}_2\mathcal{F}_{1,k}[(\mathrm{c-a,k}),(c-b,k);(c,k);z] \quad (3.4)$$

Proof: Consider

$${}_2\mathcal{F}_{1,k}[(\mathrm{a,k}),(c-b,k);(c,k);w] = (1-\mathrm{kw})^{-\frac{(c-b)}{k}}{}_2\mathcal{F}_{1,k}\left[(\mathrm{c-b,k}),(c-a,k);(c,k);-\frac{w}{(1-kw)}\right]$$

On taking

$$w = -\frac{z}{1-kz} \quad i.e. \quad z = -\frac{w}{1-kw} \textit{ and } (1-kw) = (1-kz)^{-1}$$

$${}_2\mathcal{F}_{1,k}[(\mathrm{a,k}),(c-b,k);(c,k);w] = (1-\mathrm{kz})^{\frac{(c-b)}{k}}{}_2\mathcal{F}_{1,k}[(\mathrm{c-b,k}),(c-a,k);(c,k);z]$$

$${}_2\mathcal{F}_{1,k}\left[(\mathrm{a,k}),(c-b,k);(c,k);-\frac{z}{1-kz}\right] = (1-\mathrm{kz})^{\frac{(c-b)}{k}}{}_2\mathcal{F}_{1,k}[(\mathrm{c-b,k}),(c-a,k);(c,k);z] \quad (3.5)$$

Multiplying both side of (3.5) by $(1-\mathrm{kz})^{-\frac{a}{k}}$ we get

$$(1-\mathrm{kz})^{-\frac{a}{k}}\,{}_2\mathcal{F}_{1,k}\left[(\mathrm{a,k}),(c-b,k);(c,k);-\frac{z}{1-kz}\right] = (1-\mathrm{kz})^{\frac{(c-a-b)}{k}}{}_2\mathcal{F}_{1,k}[(\mathrm{c-b,k}),(c-a,k);(c,k);z]$$

and using theorem (3.1) we get

$${}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z] = (1-\mathrm{kz})^{\frac{1}{k}(c-a-b)}{}_2\mathcal{F}_{1,k}[(\mathrm{c-a,k}),(c-b,k);(c,k);z]$$

Corollary 3.3: If we take $k = 1$ in (3.4), we have

$${}_2\mathcal{F}_1[a,b;c;z] = (1-z)^{(c-a-b)}{}_2\mathcal{F}_1[\mathrm{c-a,c-b;c;z}] \quad (3.6)$$

Which is known as Euler identity for usual hypergeometric functions.

Derivative Properties of k-hypergeometric function

Theorem 4.1: The following derivative formula for the k-hypergeometric function hold true

$$\frac{d^n}{dz^n}\{{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\}$$
$$=\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\,{}_2\mathcal{F}_{1,k}\,[(a+nk,k),(b+nk,k);(c+nk,k);z]$$
$$(k\in IR^+, n\in N_0, a,b\in\mathbb{C}, c\in\mathbb{C}/\mathbb{Z}_0^-) \quad (4.1)$$

Proof: By using (2.3) and differentiating ${}_2\mathcal{F}_{1,k}$ term by term under the sign of summation with respect to z we observe that

$$\frac{d}{dz}\{{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\}=\sum_{r=0}^{\infty}\frac{(a)_{r,k}(b)_{r,k}}{(c)_{r,k}}\frac{rz^{r-1}}{r!}$$
$$=\sum_{r=1}^{\infty}\frac{(a)_{r,k}(b)_{r,k}}{(c)_{r,k}}\frac{z^{r-1}}{(r-1)!}$$
$$=\sum_{m=0}^{\infty}\frac{(a)_{m+1,k}(b)_{m+1,k}}{(c)_{m+1,k}}\frac{z^m}{m!}$$
$$=\sum_{m=0}^{\infty}\frac{a(a+k)_{m,k}b(b+k)_{m,k}}{c(c+k)_{m,k}}\frac{z^m}{m!}$$
$$=\frac{ab}{c}\,{}_2\mathcal{F}_{1,k}\,[(a+k,k),(b+k,k);(c+k,k);z] \quad (4.2)$$

Therefore equation (4.1) is true for $n=1$

Now let equation (4.1) is true for $n=m$

$$\frac{d^m}{dz^m}\{{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\}$$
$$=\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\,{}_2\mathcal{F}_{1,k}\,[(a+mk,k),(b+mk,k);(c+mk,k);z]$$
$$=\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\sum_{r=0}^{\infty}\frac{(a+mk)_{r,k}(b+mk)_{r,k}}{(c+mk)_{r,k}}\frac{z^r}{r!} \quad (4.3)$$
$$\frac{d^{m+1}}{dz^{m+1}}\{{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\}$$
$$=\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\sum_{r=0}^{\infty}\frac{(a+mk)_{r,k}(b+mk)_{r,k}}{(c+mk)_{r,k}}\frac{rz^{r-1}}{r!}$$
$$=\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\sum_{r=1}^{\infty}\frac{(a+mk)_{r,k}(b+mk)_{r,k}}{(c+mk)_{r,k}}\frac{z^{r-1}}{(r-1)!}$$
$$=\frac{(a)_{m,k}(b)_{m,k}}{(c)_{m,k}}\sum_{\mu=0}^{\infty}\frac{(a+mk)_{\mu+1,k}(b+mk)_{\mu+1,k}}{(c+mk)_{\mu+1,k}}\frac{z^{\mu}}{\mu!}$$
$$=\frac{(a+mk)(a)_{m,k}(b+mk)(b)_{m,k}}{(c+mk)(c)_{m,k}}\times\sum_{\mu=0}^{\infty}\frac{(a+(m+1)k)_{\mu,k}(b+(m+1)k)_{\mu,k}}{(c+(m+1)k)_{\mu,k}}\frac{z^{\mu}}{\mu!}$$
$$=\frac{(a)_{m+1,k}(b)_{m+1,k}}{(c)_{m+1,k}}$$
$$\times\,{}_2\mathcal{F}_{1,k}\,[(a+(m+1)k,k),(b+(m+1)k,k);(c+(m+1)k,k);z] \quad (4.4)$$

so, the result (3.3.1) for $n=m+1$

therefore, by induction method the result (3.3.1) is true for every $n\in N_0$.

Theorem 4.2: If $a,b,c\in\mathbb{C},\Re(a)>0,\Re(b)>0,\Re(c),>0,|z|<\frac{1}{k}$, *then*

$$\frac{d^n}{dz^n}\left\{z^{\frac{a}{k}+n-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\frac{1}{k^n}(a)_{n,k}\,z^{\frac{a}{k}-1}\,{}_2\mathcal{F}_{1,k}\,[(a+kn,k),(b,k);(c,k);z] \quad (4.5)$$

Proof:

$$\frac{d}{dz}\left\{z^{\frac{a}{k}+n-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}=\frac{d}{dz}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^{\frac{a}{k}+2n-1}}{n!}\right\} \quad (4.6)$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\left(\frac{a}{k}+2n-1\right)\frac{z^{\frac{a}{k}+2n-2}}{n!}$$
$$\frac{d^2}{dz^2}\left\{z^{\frac{a}{k}+n-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\left(\frac{a}{k}+2n-1\right)\left(\frac{a}{k}+2n-2\right)\frac{z^{\frac{a}{k}+2n-3}}{n!} \quad (4.7)$$

Now, n repeated applications of the derivative formula for the function ${}_2\mathcal{F}_{1,k}$ yields

$$\frac{d^n}{dz^n}\left\{z^{\frac{a}{k}+n-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\left(\frac{a}{k}+2n-1\right)\left(\frac{a}{k}+2n-2\right)\ldots\ldots\left(\frac{a}{k}+n\right)\frac{z^{\frac{a}{k}+n-1}}{n!}$$
$$=z^{\frac{a}{k}-1}\sum_{n=0}^{\infty}\frac{1}{k^n}\frac{(a)_{n,k}(b)_{n,k}(a+kn)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!}$$
$$=\frac{1}{k^n}(a)_{n,k}\,z^{\frac{a}{k}-1}\,{}_2\mathcal{F}_{1,k}\,[(a+kn,k),(b,k);(c,k);z]$$

Theorem 4.3: If $a,b,c\in\mathbb{C},\Re(a)>0,\Re(b)>0,\Re(c),>0,|z|<\frac{1}{k}$, then

$$\frac{d^n}{dz^n}\left\{z^{\frac{c}{k}-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\frac{1}{k^n}(c-kn)_{n,k}\,z^{\frac{c}{k}-n-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c-nk,k);z] \quad (4.8)$$

Proof:

$$\frac{d}{dz}\left\{z^{\frac{c}{k}-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}=\frac{d}{dz}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^{\frac{c}{k}+n-1}}{n!}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\left(\frac{c}{k}+n-1\right)\frac{z^{\frac{c}{k}+n-2}}{n!} \quad (4.9)$$
$$\frac{d^2}{dz^2}\left\{z^{\frac{c}{k}-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\left(\frac{c}{k}+n-1\right)\left(\frac{c}{k}+n-2\right)\frac{z^{\frac{c}{k}+n-3}}{n!} \quad (4.10)$$

Now, n repeated applications of the derivative formula for the function ${}_2\mathcal{F}_{1,k}$ yields

$$\frac{d^n}{dz^n}\left\{z^{\frac{c}{k}-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\left(\frac{c}{k}+n-1\right)\left(\frac{c}{k}+n-2\right)\ldots\ldots\left(\frac{c}{k}\right)\frac{z^{\frac{c}{k}+n-(n+1)}}{n!}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{1}{k^n}(c+k(n-1))(c+k(n-2))\ldots\ldots(c)\frac{z^n z^{\frac{c}{k}-(n+1)}}{n!}$$
$$=\frac{(c-nk)_{n,k}}{k^n}\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c-nk)_{n,k}}\frac{z^n z^{\frac{c}{k}-(n+1)}}{n!}$$
$$=\frac{1}{k^n}(c-kn)_{n,k}\,z^{\frac{c}{k}-n-1}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c-nk,k);z]$$

Theorem 4.4: *If* $a,b,c\in\mathbb{C},\Re(a)>0,\Re(b)>0,\Re(c),>0,|z|<\frac{1}{k}$, then

$$\frac{d^n}{dz^n}\left\{z^{\frac{c}{k}+n-1}(1-kz)^{\frac{1}{k}(a+b-c+nk)}\,{}_2\mathcal{F}_{1,k}[(a+nk,k),(b+nk,k);(c+nk,k);z]\right\}$$
$$=\frac{1}{k^n}(c)_{n,k}\,z^{\frac{c}{k}-1}\,(1-kz)^{\frac{1}{k}(a+b-c)}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z] \quad (4.11)$$

Proof: On replacing $a,b\ and\ c\ to\ a+nk,b+nk\ and\ c+nk$ respectively in theorem (3.2.2), we get

$${}_2\mathcal{F}_{1,k}\,[(a+nk,k),(b+nk,k);(c+nk,k);z]$$
$$=(1-\mathrm{k}z)^{\frac{1}{k}(c-a-b-nk)}\,{}_2\mathcal{F}_{1,k}[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c+nk,k);z] \quad (4.12)$$
$$\frac{d^n}{dz^n}\left\{z^{\frac{c}{k}+n-1}(1-kz)^{\frac{1}{k}(a+b-c+nk)}\,{}_2\mathcal{F}_{1,k}\,[(a+nk,k),(b+nk,k);(c+nk,k);z]\right\}$$
$$=\frac{d^n}{dz^n}\left\{z^{\frac{c}{k}+n-1}\,{}_2\mathcal{F}_{1,k}[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c+nk,k);z]\right\}$$
$$=\frac{d^n}{dz^n}\left\{z^{\frac{C}{k}-1}\,{}_2\mathcal{F}_{1,k}[(\mathrm{A},\mathrm{k}),(B,k);(C,k);z\right\}$$

where $A=c-a, B=c-b, C=c+nk$

$$=\frac{1}{k^n}(C-kn)_{n,k}\,z^{\frac{C}{k}-n-1}\,{}_2\mathcal{F}_{1,k}[(\mathrm{A},\mathrm{k}),(B,k);(C-nk,k);z]$$

using theorem (4.3)

$$=\frac{1}{k^n}(c)_{n,k}\,z^{\frac{c}{k}-1}\,{}_2\mathcal{F}_{1,k}[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c,k);z]$$
$$=\frac{1}{k^n}(c)_{n,k}\,z^{\frac{c}{k}-1}(1-kz)^{\frac{1}{k}(a+b-c)}\,{}_2\mathcal{F}_{1,k}[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z]$$

using theorem (3.2)

Theorem 4.5:

$$\frac{d^n}{dz^n}\left\{z^{\frac{1}{k}(c-a)+n-1}(1-kz)^{\frac{1}{k}(a+b-c)}\,{}_2\mathcal{F}_{1,k}\ [(a,k),(b,k);(c,k);z]\right\}$$
$$=\frac{1}{k^n}(c-a)_{n,k}\,z^{\frac{1}{k}(c-a)-1}(1-kz)^{\frac{1}{k}(a+b-c-nk)}$$
$$\times\,{}_2\mathcal{F}_{1,k}\,[(a-nk,k),(b,k);(c,k);z] \tag{4.13}$$

Proof:

$$\frac{d^n}{dz^n}\left\{z^{\frac{1}{k}(c-a)+n-1}(1-kz)^{\frac{1}{k}(a+b-c)}\,{}_2\mathcal{F}_{1,k}\,[(a,k),(b,k);(c,k);z]\right\}$$
$$=\frac{d^n}{dz^n}\left\{z^{\frac{1}{k}(c-a)-1}\,{}_2\mathcal{F}_{1,k}\,[(c-a,k),(c-b,k);(c,k);z]\right\}$$

using theorem (3.2)

$$=\frac{1}{k^n}(c-a)_{n,k}\,z^{\frac{1}{k}(c-a)-1}\,{}_2\mathcal{F}_{1,k}[(c-(a-nk),k),(c-b,k);(c,k);z]$$

using theorem (3.1)

$$=\frac{1}{k^n}(c-a)_{n,k}\,z^{\frac{1}{k}(c-a)-1}(1-kz)^{\frac{1}{k}(a+b-c-nk)}$$
$$\times\,{}_2\mathcal{F}_{1,k}\,[(a-nk,k),(b,k);(c,k);z]$$

Conclusion

In this paper, we have investigated useful relations and formulae involving the k-hypergeometric functions. In section (3), we have proved the transformation formulae involving the above functions and deduced the Euler’s identity for usual hypergeometric functions, which is useful in deduction of Kummer’s twenty-four solution of Gauss hypergeometric differential equation. The properties associated with the analyticity of the k-hypergeometric functions have been proved in the next section.

The k-hypergeometric functions, which is an extension of hypergeometric functions, include as a special case, most of the commonly used special functions. A number of problems of mathematical physics and engineering are capable of being represented in terms of this functions; hence the obtained results could be useful in more complex problems in physical sciences.

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